STUDY OF DISCRETE TIME ANALYSIS AND ESTIMATION OF MARKOVIAN TANDEM QUEUEING MODELS

THESIS

ESTB. AUGUST 26, 1975

Submitted To
BUNDELKHAND UNIVERSITY, JHANSI
For the Award of the Degree of
Doctor of Philosophy
In
MATHEMATICS

By
Narendra Pathak

Under the Supervision & Guidance of
Prof. Vijay Kumar Sehgal
Director & Head
Dean, Faculty of Science

DEPARTMENT OF MATHEMATICAL SCIENCE & COMPUTER APPLICATION
BUNDELKHAND UNIVERSITY
JHANSI - 284128

2007

SUPERVISOR'S CERTIFICATE

This is to certify that the thesis entitled " **Study of Discrete Time Analysis and Estimation of Markovian Tandem Queueing Models.**" Which is being submitted by **"Narendra Pathak"** for the award of Ph.D. degree in mathematics to the Bundelkhand University, Jhansi is a record of bonafide research work carried out by him under my supervision and guidance. The thesis embodies the work done by the candidate himself and has not been submitted to any other University/Institution for the award of any degree/diploma to the best of my knowledge further he has completed the required attendance in this duration as per University rules and norms.

Sehgal
28/9/07

Prof. V. K. SEHGAL
(Supervior)
Director and Head
Dean, Faculty of Science
Deptt. of Mathematical Science
and Computer Application
Bundelkhand University, Jhansi

DECLARATION

I hereby declare that the thesis entitled **"Study of discrete time analysis and estimation of Markovian tandem queueing models"** being submitted for the degree of Doctor of Philosophy in Mathematics, Bundelkhand University, Jhansi (U.P.), is an original piece of research work done by me under the supervision of **Prof. V.K. Sehgal**, Director & Head, Department of Mathematical Science & Computer Applications, Bundelkhand University, Jhansi (U.P.) and the best of my knowledge and belief is not same as the one which has already been submitted for a degree of any academic qualification at any University or examining body in India or abroad.

Pathak

(NARENDRA PATHAK)

Researcher

ACKNOWLEDGEMENT

No words can adequately express the sense of gratitude that I owe to my supervisor, **Professor V. K. Sehgal** who has been a source of constant inspiration and affectionate guidance. It was really a pleasure of great deal under him because of his frank discussions and painstaking efforts. Needless to add, this work has been completed because of his very able and valuable guidance and working under him was a rewarding experience.

My greatful thanks are due to Professor Dr. P. K. Kapur, Director, Deptt. Of Operation Research, University of Delhi and Dr. A. K. Bardhan, Delhi University. Who has supported me and gave me all help and encourgement.

I am also thankful to Dr. J.P.N. Mishra, Principal Dr. R.P.Richhariya P.G. College for all help and encouragement.

Last. But not the least. I sincerely acknowledge blessing showered on me by my parents and good wishes to me from my all family members.

NPathak

(NARENDRA PATHAK)

Lecturer, Deptt. of Mathematics

Dr. R.P. Richhariya P.G. College

Baruasagar, Jhansi (U.P.)

CONTENTS

CHAPTER - 1
INTRODUCTION

CHAPTER - ONE

INTRODUCTION

Operational research is the application of the methods of science to complex problems arising in the direction and management of large system of men, machines, materials and money in industry business, government and defence. The distinctive approach is to develop a scientific model of the system in corporation measurement of the factors such as chance and risk. With which to predict and compare the outcome, alternating decisions, strategies, control. The purpose is to help management to determine its policy and action scientifically. Operational Research is concerned with scientifically deciding how to best design men machine system. Usually under condition requiring the allocation of scarce resources. Operational research can be extremely useful in solving problems in developing countries as well as developed countries. The developed countries if confronted with difficult problem can afford in some money in projects money and resources in projects of obscure value, but this is not so far the developing courtiers. As they have to be extremely careful in utilizing their resources for getting the optimum results which needs the technique of operational Research. The essence of operational research activity lies is the construction and use of model.

A model is a simplified representation of something result. There are many conceivable reason why one might prefer to deal with a substitute for the real thing rather than with the thing itself. Often motivation is to save the money, time and some other valuable commodity. Some time it is to wide risk associate it with tempering of with real objects. Sometimes the real environment is so complicated that a represantive model is

needed just to understand and to communicate with others about it. Queueing analysis has a prominent place among various branches of operational research as it deals with phenomena that we all encounter as an everyday lives. In fact one is almost encounter some form of queue or waiting line in every waking hour. For example such as standing in line in ticketing window. There are many subtle cases of queuing behaviors that we experience further while making a telephone call. There is a brief delay as a free communication channel is to be located while driving a car, a traffic congestion encountered on the open highway of the Queueing behavious. The flow of the material through a manufacturing in various processing stages represent other form of queue. Queues are found throughout business in administrative office, medical science, hospital administration, chemical industries and in communication technologies. Consider the following example of queues of a food processing company may have truck loaded with perishables lined up for over 24 hours waiting to be unloaded not only spoilages quite only possible perishables already loaded. But also those for fields waiting to be loaded of on returning trucks. Another queue is found in business frequently associative with inventory. In administration office the paper flows throughout the offices and it will be recognized that queue can block this communication system. Queue also exist in physical being specific genetic defects in born errors in metabolism in one such disorder a missing principle enzyme cause a substance to accumulate wait in queue in blood stream. This eventually damages the brain. Scientist in studying the cause and cure of such diseases as cancer and leukemia have found that the breaking down of all these process leads to the queue. That makes the disease system.

In today world it is almost impossible to point out the area in which queue does not exist because in the advancement in science and technology has increased the dimension of

the man and society as a whole to great extent where as the resource are finite and which are to be taken into account and therefore there are congestion situation. Queueing theory found almost all area demonstrating the divergent nature of the problem. It is applicable to inventories best office backlog, large backlog in our court system, catalyst in chemical reaction, filtration process, semiconductor noise hospital and the demand for medical care nervous reaction such as psychological stress and strain neurosis production line, inspection line conveyor built. Then there are those barber shop if all short utilities mass transportations and communication and so many other facilities. So the life present a very large number of queues some are plants or some unexplored consider the amount of time wasted while waiting one week, one month or one year period. Hence queues are important to managed properly because it is a problem us all directly or indirectly when designing a new system or revamping the old one engineers and businessman must be cognizant of inherent line. Super market must be planned for adequate check out system and servicing system as dissatisfaction may restrict buying or loss of customers. Hospital administrators and architects often engage in facilities planning for facilities. Waiting line is common place in hospital and they create emotional hardships for the patients waiting for foods, admittance, surgery and laboratory work etc. This creates overall frustration, which is to be tackled sympathetically. In chemical industry which work on constantly with filtration problems must provide that the particles that are gathered by the filter can be cleared at a rate greater than the partial arrival rate otherwise the filter will clog and cause the whole system to be idled.

Queueing theory is a mathematical study of waiting line. A basic Queueing model has two distinct part.

1. Waiting line.
2. Service facility

An incorrect design or operating philosophy will magnify all problem areas associated with waiting. Thus the study of Queueing theory must incorporate both physical and cost characteristics associated with various system. Queues are frequently encountered in various networks such as telecommunication, production jobs shops, flexible manufacturing process, computer system, urban services system, communication networks, health care system, and transportation system.

A network is a system of lines or channels connecting different points. Some example of network are communication line, rail board network, pipeline system, shipping line and aviation networks and all those networks some specified commodities is sent from certain supply point to certain one because all this networks having multiple resources to be taken into the account. For example computer network deal with C.P.U., channels memories, communication circuits etc. A queues unavoidable at these facilities these complex structure leads to a Queueing networks rather than a single queue with single server. Today the information processing industry is one of the fastest growing, most dynamic and most glamoures industry on the scientific seen. It highly depends on the enormously grown complex networks. The structure and sophistication of these networks varies over a considerable range from highly specialized networks design to handle to specific task. In a carefully environment to more generalized networks that handle a variety of task in highly structured environment.

Queueing networks are also classified according to service time distribution either being exponential or general service time. All customers may belong to the same class with the same switching process and requesting similar services or there may be different types of costumers. Many practical systems can be modeled of queues open queuing networks

accepts and losses costumers from to the out side world. Thus the total number of costumer in open networks varies with time.

TANDEM – QUEUES

Queueing system with several service facilitation in service is known as tandem Queueing system. This type of Queueing networks can represent many process of interest in manufacturing system, computer system tale communication etc. The communication system can be typically modeled as networks of interconnected queues by viewing the massage as customer communication buffer as waiting line and all activities in necessary transmission of the message as services.

This sort of synchronization has advantages of conceptual simplicity and great generality. This leads communication network to view as tandem Queueing system. In the tandem Queueing networks there might be an infinite or finite queue capacity at each service unit of the network. The tandem queue can be representative for modeling a practical multi stage service system in manufacturing process and communication networks.

It is important to study these queues, for the queues in these areas effect the efficiency of the system concerned and cause a huge wastage of precious resources, which is ought to be reduced by optimum utilization of resources.

The production system consists of a number of service facilities where item receive service. These system can be very simple or can be very complex with several machines linked together in an arbitrary manner, with unequal processing times, limited space for temporary storage of items and general routing patterns for different jobs going through the

system. This is especially true in flexible manufacturing system. These system can also be represented as tandem queues by representing the machines as service nodes and the item as customer.

1.1 REVIEW OF LITERATURE

The modelling of computer system and data transmission system opened the way, in the sixties, to studies of queues characterized by complex service disciplines and have created the need of analyze inter connected systems progress in this area has been rapid and industrial application have been widely accepted since the seventies at present in the computer industry, Queueing network models have resulted in software packages for the automatic solution of problems arising in the design of new computer systems in the evaluation and improvement of existing system.

Tandem queues can be representative for modelling practical multistage service system in manufacturing process and communicational systems, which attracted many research workers to study tandem queue and a lot of work has been done on this line to present a brief historic account fallowing works are worth mentioning.

O.Brien, G.G. (1954) studied the case of two channels in series in the steady state with Poisson input and exponential service times and gave expected number and the expected waiting times. Burke, P. J. et.al (1956) found the output of a steady state queue, independent of the type of service discipline used. Jackson, J.R. (1957), both permitted to Poisson arrival to each phase, both from the system and from out side. The service distributions are exponential and each phase consists of several parallel channels. An important questions to determine in studying tandem queues in the distribution of out put from one channel, which the comparises the input into the subsequent channel Reich (1957) discussed this problem. Finch, P.D. (1959) shown that it is only when an infinite size queue is tolerated that the Poisson result for the output is current. He also gave that $N=\infty$ and exponential service as a necessary and sufficient condition for the independence of the

interdeparture intervals and the independence of the queue length left by a departing unit from the interval since the previous departure.

Friedman (1965) worked on reduction methods for tandem queues. Hillier and Boling (1967) studied the tandem (or serial) networks with exponential servers, the most basic structural configuration. Neuts (1968) considered two queues in series with a finite intermediate waiting room. Fraker (1971) gave approximate techniques for the analysis of tandem Queueing system. Tembe et.al (1974) provided the optimal order of service in tandem queues. Kobayashi (1974) applied diffusion appromimations to Queueing networks. Chandy et.al (1974) did approximate analysis of general Queueing network. Rubin (1974) provided path delays in communication networks. Baskett et.al (1975) studied open, closed and mixed networks of queues with different classes of customers, Lavenberg (1975) discussed stability and maximum departure rate of certain open queuing networks having finite capacity constraints. Gelenbe et.al (1976) studied the behaviour of a single queue in a general Queueing network Schweitzer (1976) obtained maximum through put in finite capacity open queuing network. Kleinrock (1878) studied the communication networks as tandem queue. They considered the independent assumption among the service and arrival processes. Labetoulle et.al (1977) modelled packed switching communication networks with finite buffer size at each node. Dattatreya (1978) shown that the through put of a given system is the same as that of the reversed system, and studied tandem queuing system with blocking. Harrison (1978) applied diffusion approximation for tandem queues in heavy traffic.

Chandy et.al (1978) gave approximate methods for analyzing Queueing network of computing systems. Weber (1979) discussed the interchangeability of tandem/M/1 queues

in series. Boxma (1979) worked a tandem Queueing model with identical service times at both counties. Reiser (1979) did a queuing network analysis of computer communication network with window flow control. Kuehn (1979) analysed networks approximately by de-composition. Strelen et.al (1979) analyzed Queueing networks with blocking using a new aggregation technique. Takahashi et.al (1980) considered the general system, which is a combination of tandem, split and merge configurations which is the most complicated to analyze, assuming that effective service times follows an exponential distribution, they developed a set of simultaneous nonlinear equations that must be solved to get performance measures. Ammar (1980) did not modeling and analysis of unreliable manufacturing assembly networks with finite storages Latouche et.al (1980) provided efficient algorithmic solutions to exponential tandwm queues with blocking. Bure (1980) discussed the dependence of sojourn times in tandem m/m/s queues. Seraphin (1981) made considerable efforts in developing and analyzing the interdependent communication networks composed of more than one transmission line connected in tandem.

Ammar et.al (1981) gave equivalence relations in Queueing models of manufacturing. Shanti Kumar et.al (1981) provided open queueing network models of dynamic job shops internate. Ohno (1981) worked on a test problem that has a large number of states in order to evaluate algorithms for solving discounted semi-markov decision processes and compare several variants of there algorithms. Perros (1981) analyzed a symmetrical exponential open queue network with blocking and feed back. Sauer (1981) gave an approximate solution for Queueing networks with simultaneous resource possession. Pinedo (1982) worked on the optimal order of stations tandem queues. Wolff

(1982) discussed tandem queues with dependent service in light traffic. Fakinas (1982) studied the generalized M/G/K blocking system with heterogeneous servers.

Altiok (1982) gave an approximate analysis of exponential tandem queues with blocking Bouanaka (1982) discussed approximating queueing networks and facility planning. Gershwin et.al(1983) did modelling and analysis of three stage transfer line with unreliable machines and finite buffers. Whitt (1983) worked on the Queueing networks analyzer. Reiman (1984) discussed an open Queueing networks in heavy traffic. Coffman et.al (1984) used diffusion approximation for computer communication system. Jenq (1984) gave approximations for packetized voice in statistical multiplexer in process. Ganz O. (1984) provided an analysis of multiple bus synchronous and asynchronous communication system. Langaris et.al (1985) studied three stage tandem queue with blocking. Flores (1985) applied diffusion approximations for computer communication networks. Marshal et.al (1985) derived statistical measures for mixed date traffic on a local area network Yao (1985) did modelling for a class of state dependent routing inflexible manufacturing systems. Disney et.al (1985) reviewed queueing networks and gave survey of their random process Altiok et.al (1985) did approximate analysis of arbitrary configurations of open Queueing networks with blocking. Carmelila (1985) worked for improving mean delay in data communication networks by new combined strategies based in the soft principle. Pollok et.al (1985) gave an approximation method for tandem queues with blocking with exponential and general service distributions. Brandwajn et.al (1985) also gave an approximate method for tandem queues with blocking due to finite waiting room. Harrison (1985) worked on normalizing constants in Queueing networks. Bermanoded. et.al (1985) studied about optimal server location on a networks operating as an M/G/1 queue.

Greenberg (1986) discussed queueing systems with returning customers and the order of tandem queues. Perros et.al (1986) gave an approximate analysis of open networks of queues with blocking having tandem configuration. Vinod et.al (1986) worked for approximating unreliable Queueing networks under the assumption of exponentially. Buzzacott et.al (1986) considered Queueing networks models of flexible manufacturing system.

Altiok (1986) studied open networks of queues with blocking split and merge confiquration. Kerbache (1986) provided the generalized expansion method for open finite Queueing networks. Perros et.al (1986) discussed a computationally efficient approximation algorithms for analyzing open Queueing networks with blocking and having tandem configuration. Gershwin (1987) gave an efficient decomposition method for the approximate evaluation of tandem queues with finite storage space and blocking.

Tsoucas et.al (1987) discussed the interchangability and stochastic ordering of /M/1 queues in tandem. Shanti Kumar et.al (1987) derived bounding performance of Tandem queues with finite buffer spaces. Harrison et.al (1987) worked on Brownian models of open Queueing networks with homogeneous customer populations Altiok, et.al (1987) gave an approximate analysis of arbitrary configuration of open Queueing networks with blocking. Kerbache et.al (1987) provided the generalized expansion method for open finite Queueing networks. Schmidt (1987) found joint queue length characteristics in infinite server tandem queues with heavy traffic.

Ohno (1987) computed optimal policies for controlled tandem Queueing systems. Brandwajn et.al (1988) gave an approximation method for tandem queues with bloking Vandijk (1988) obtained bounds for call congestion of finite single server exponential

tandem queues, Helber (1988) did decomposition of unreliable assembly, disassembly networks with limited buffer capacity and random processing times, Perros et.al (1988) provided approximate analysis of product form type Queueing networks with blocking and dead lock.

Bitran et.al (1988) discussed multiproduct Queueing networks with deterministic routing using decomposition approach and the notion of interference. Kerbache, et.al (1988) studied asymptotic behaviour of the expansion method for open finite Queueing networks.

King (1988) studied Queueing models for buffer with dial up servers. Gun, et.al (1989) provided an approximation method for general tandem Queueing systems subjects to blocking. Perros, et.al (1989) gave a computationally efficient approximation algorithm for feed forward open Queueing networks with blocking. Akylildiz, et.al (1989) found computational algorithms for networks of queues with rejection blocking. Harrison et.al (1989) did heavy traffic analysis of a simple open networks. Foss, (1989) derived some properties of open queuing networks. Brandwajn, et.al (1989) gave a note on approximate interactive solution of open tandem networks with blocking Lee, et.al (1989) gave an approximate analysis for the merge configuration of an open Queueing networks with blocking Vandijk, et.al (1989) provided simple bound and monotonicity results for multiserver exponential tandem queues. Van Dijk (1989) obtained simple bounds and monotonicity results for multiserver exponential tandem queues. Jon and Perros (1990) provided an approximate analysis for open tandem Queueing networks with blocking and general service times.

Harrison et.al (1990) used the QNET method for two moment analysis of open queueing networks. Sigman (1990) discuss the stability of open queueing networks. Dallery (1990) gave an approximate analysis of generalized open Queueing networks with restricted capacity. Harry and Perros (1990) provided approximation algorithms for open Queueing networks with blocking.

Christopher (1990) worked on diffusion approximations for computer communication networks. Gupta et.al (1990) considered production systems with two job classes and change over times and revisitation. Peterson (1991) worked on diffusion approximations for networks of queues with multiple customer types.

Rybko, el.al (1991) discussed ergodicity of stochastic processes describing the operation of an open Queueing networks fouler and lelond (1991) studied local area network traffic characteristics with implications for broad band networks congestion management. Horvath (1992) gave strong approximations for open Queueing networks. Kella, et.al (1992) took a tandem fluid networks with levy input, in Queueing and related models. Bitran and Dasu (1992) reviewed open Queueing networks models of manuifacturing system. Dallery and Gershwin (1992) reviewed models and analytical results for manufacturing flow line systems. Jungshyrwiv (1992) did maximum entropy analysis of open Queueing networks with group arrival. Xinlichao, (1992) discussed reversibility of tandem queues with blocking. Iliadis (1992) studied synchronous versus asynchronous operation of a packet switch with combined input and output Queueing. Yakuo Hyshida (1993) did through put analysis of tandem type goback-NARQ, scheme for satellite communications. Dallery and Frain (1993) discussed decomposition methods for

tandem Queueing networks with blocking. Kelly and Laws (1993) considered dynamic routing in open Queueing networks.

Harrison, et.al. (1993) gave a Brownian models of multi class Queueing networks. Spiros and Dnozis (1993) worked on entropy maximized Queueing networks with blocking and multiple job classes. Reyravi, Hassan (1993) derived limiteg distributions of the blocking propability for circuit switched networks. Park, et.al (1994) gave an approximate analysis of discrete time tandem networks with busty and correlated input traffic and customer loss.

Waller (1994) provided a Queueing network models for field service support system. XINLTCHAO, (1994) considered a priority tandem queue with no intermediate buffer, Jain and Smith (1994) studied open finite Queueing networks with M/M/CK parallel servers. Yannopoul as and Alfa (1994) gave a simple and quick approximation algorithm for tandem split and merge Queueing networks with blocking.

Sakasegawa and Kawashima (1994) studied about the equivalence of three types of blocking in non-markovian tandem queues. Balsamo, el.al (1994) did survey of product form Queueing networks with blocking and their equivalences. Frostand and Melamed (1994) did traffic modeling for telecommunications networks. Erramilli, et.al (1994) studied fractal traffic flows in high-speed communication networks. Avram et.al (1995) used optimal control approach for fluid models of sequencing problems in open Queueing networks.

Dai (1995) discussed stability of open multi class Queueing networks via fluid modles. Nguyen (1995) used fluid and diffusion approximations of a two station mixed Queueing networks Pots, G. (1995) studied state dependent queuing networks. Liu and Lin

(1995) gave an efficient two-phase approximation method for exponential tandem Queueing systems with blocking. Gerasimov and Alxander (1995) worked on normalizing constants in multiclass Queueing networks.

Vijaya Kumar (1996) studied some interdependent tandem Queueing models with applications to communications networks. Chakka and Mitrani (1996) gave approximation solution for open networks with break down and repairs. Gansnoah and Ryzinvangarrett, (1997) discussed about optimal control of a multi class flexible Queueing system. Lee, et.al (1998) worked on performance evaluation of open Queueing network with arbitrary configuration and finite buffer. Skianis, et.al (1998) studied arbitrary open Queueing networks with server vacation periods and blocking.

Williams (1998) used diffusion approximations for open multiclass Queueing networks Dai, et.al (1999) proved heavy traffic limit theorem for a class of open Queueing networks with finite buffers. Knessi and Tier (1999) considered two tandem queues wit general renewal input using. Diffusion approximation and integral representation. Daduna and Szekli (2000) discussed the correlations of sojorn times in open networks of exponential multiserver queues. Grassmann and Dekic (2000) gave an analytical solution for a tandem queue with blocking. Victar and Ivnitski (2001) considered networks of single server queues with dependent service times.

1.2 BROAD OUTLINES OF THE WORK

Murty (1993) extended the single server interdependent queueing model to bulk service models $M/M^{[x]}/1$. He considered a different problem which is a generalization of the earlier Queueing model. Two variations are considered in the first one, the customer are served K at a time except, when less than K are in the system and ready for service at which time all units are served and in the second one the batch size must be exactly K and if it is not the server remains idle until the queue, size reaches K, a typical situation of this model is transportation of men and material are the customer. For this type of situation with bulk service rule they make use of along with the other assumption he assumed that batch size must the dependence structure given by Rao (1986). Further, Murty (1993) considered a slight variation of the model be exactly K and if not the server waits until such time to start. In order to reduce the idle time of the server it is feasible to consider the interdependence in the model also. He further assume that the number of arrivals and the service of any branch are correlated and follows a bivariate Poisson process.

In the bulk service queueing models $M/M^{[x]}/1$, Bailey (1954) and Jaiswal (1960) considered models in which the units arrive at random from a single queue in order of arrival and are used in batched, the size of each batch being either a fixed number of customers or the whole queue length which ever is smaller. Jaiswal (1961) extended this model to the case where at a service epoch m ($0 \leq M \leq S$) if m persons are already present with the server then (S-m) persons or the whole queue length which is smaller will we taken in to service. The service rule is termed as Bailey's is bulk service. However, in these models the arrivals and service process are poissonian and erlangian respectively with interdependent arrival and service processes. In both the models the system behaviour is

analyzed by obtaining the difference differential equation of the model and solving them through generating function techniques. The system characteristic like mean queue length, variability of the system size, coefficient of variations are derived and analyzed in the light of the dependence parameter. These models also include the earlier models as particular cases for specific values of the parameters. Murty (1993) considered the single server system with interdependent arrival and service process having the Bailey's bulk service with variable capacity. Here he assumed that the server serves only at instants. $t_1, t_2 \ldots\ldots\ldots\ldots t_n \ldots\ldots\ldots\ldots\ldots$ if m persons ($0 \leq M \leq S$) are present in the waiting line at time t_n then the server takes a batch of (S-m) persons or whole queue length which ever is smaller, where s is the service capacity. He also considered two server inter-dependent Queueing model with Baily's bulk service rule for analysis. This model is an extension of the models considered earlier to server case it is assumed that the bulk service process having two service facilities with capacities b_1 and b_2 A batch of b_1 units or the whole queue length which ever is smaller is taken from the head of the queue for service in the first channel when even it is free. Similarly, the second channel on becoming free takes b_2 ($<b_1$) or the whole queue length which ever is less. If both the server are idle and there is no queue the next unit to arrive always goes to the first service facility. Further he assumed that the two service facilities are independent of each other and the arrival and service process are independent. This sort of situation are common on marshalling yards with two engines, elevator process with two lifts etc. This model is also extended to multi server inter – dependent Queueing model with Bailey's bulk service. In both the models he first develop the difference – differential equations and solve them through generating function technique. The system characteristics are derived analyzed in the presence of the dependent

parameter. The estimation of the parameters involved has not been consider which we propose to study in the present study and analyze the model with numerical idea.

The work done on the discrete time transient solution for some Queueing models like multiserver queue with balking and reneging first passage time distribution, double ended queue, Busy period analysis queues with heterogeneous server and machine interference model etc. has been considered.

Several Queueing problems have been solved using steady state conditions. As compared to those problems, it seems that not much have been done to obtain the corresponding transient solutions this is because of the fact that the transient solutions are not only mathematically intractable or excessively laborious but also computationally very costly. Therefore, we can say that most of Queueing theory results has concentrated on steady solution or some approximation. In most of the cases even steady state solution are difficult to compute.

Chaudhary, Kapur et.al (1991,1992) have set a new trend in the numerical computations of models in queue through the technique of using roots. Closed form solution as well as exact computation results are obtained by this approach. Takac's (1962) given two solution for $M/M/1/\infty$ neither of which is easy to compute.

The first solution is in terms intervalas where as the second involves and infinite sum of Bessel fuctions. The solution becomes a bit simpler if the waiting space is finite which may be true in many applications. In their case arrival and service rates are constant. Besides this they make use of spectral decomposition which require to find left and right eigen vectors. This is not easy if the matrix is large. The method developed by Chaudhry, Kapur et.al (1991) avoids spectral decomposition and well suited method for small and

large matrices. Numerical technique such as Runga- Kutta, Euler, Taylor and Randomisation have been used for finding transient solution- whereas the first there have been employed in solving differential equation the later on is particular suited for solving Queueing problems. In order to get greater accuracy one needs to increases to number of steps. Recently , Sharma and Das (1988) have obtained t_4 ansient solution to a special category of Markovian model using eigen values of matrices in Queueing theory. Standard numerical packages are used to obtain eigen values, which are difficult to obtain when the matrices are large and as a result computational difficulties arises. Chaudhary et.al (1991) have made attempt to obtain similar results is discrete time for finite waiting space problems in Queueing theory since the transient solution depend on the initial state of the system, it is interesting to know the effect on the system behaviour. Kobayshi (1983) has discussed the several systems which operate at discrete times of processor and several other examples in computer science. Chaudhry et.al (1991) give the transient solutions for general class of discrete times models in Queueing theory. In this way they assumed that the queue consists of finite waiting space. Inter arrival and service probabilities are dependent on the state of the system, the inter arrival and service time are geometric by independent of time and queue discipline is first in first out (FIFO). They obtained difference differential equations under these assumptions and solved these equations by the method of probability generating function. Cramer Rule has been applied for finding the solution of these equations. Explicit closed form expressions for distributions has been obtained in terms of the root of a characteristic equation. To find the eigen values or characteristic roots they make use of QROOT software package which is developed at Royal Military College at Canada by Chaudhry (1992).

Kapur, Sehgal el.al (1996) has obtained discrete time transient solution for the first passage time distribution under arbitrary initial conditions and finite waiting space. Further they worked on numerical computation of discrete time multi – server queue with balking and reneging. Although continuous time models are particular cases of discrete time models, this areas of research has remained neglected. Earlier attempts have largely concentrated on steady state solution of discrete time queues [Neuts (1989)]. It is in this sense that the study of discrete time models should stimulate to other areas such as computer science. Kapur, Sehgal et.al (1996) has further obtained discrete time transient solution for busy period analysis and double ended queuing system. They obtained closed form transient solution for the double ended Queueing system in discrete time. It is also further shown how the results in corresponding continuous time can be obtained further Kapur, Sehgal et. Al. (1996) has obtained a discrete time transient solution for Geom (N)/ Geom (N)/1/N machine interference model.

It has been also discussed multi-operating system for the machine interference model with the and arbitrary initial number of machines interference. Extensive numerical results has also been given using the closed form expressions for multi – server machines interference model. The analytic and computational aspects of the model $M^{[x]}/G/1$ and their applications has been discussed by many author. The results have been discussed by many authors. The results have been reported in transform forms. The numerical aspects of Queueing models is at least as important as their analytical development.

The work on discrete treatment solution for the first passage time multi server Queueing with bulking reneging busy period analysis has been obtained by Kapur P.K., Garg, R.B., Sehgal V.K. (1995) give numerical and transient solution of K out N:G systems

in discrete time with heterogeneous repair facilities. Kapur P.K., Sehgal V.K. , Garg R.B., Jha A.K. (1995) give numerical and transient solution of K out N:G system in discrete time with multi repair facilities.

1.3-DISCRETE TIME TRANSIENT SOLUTION

Several queuing problems have been solved using steady state conditions. As compared to these problems it seems that not much have been done to obtain the corresponding transient solution. This is because of the fact that transient solutions are not only mathematically intractable or excessively laborious but also computationally very costly. Therefore. we can say that most of queueing theory results has concentrated on steady state solution or some approximation. In most of the cases even steady state solution are difficult to compute. Chaudhry M. L., Kapur, P.K., and Templeton, J.G.C.(1991 ,1992) have set a new trend in the numerical computations of models in queue through the technique of using roots. Closed form solutions as well as exact computational results are obtained by this approach.

Tackac's (1962)gives two solutions for the $m/m/1/\infty$ neither of which easy to compute. The first solutions is in terms of integrals whereas the second involes on infinite some of bessel functions.

The solutions becomes a bit simpler if the waiting space is finite which may be true in any application. In their case arrival and service rates are constant besides this they make use of spectral decomposition which require to find left and right eigen vectors. This is not easy if the matrix is very large.

The method developed by Chaudhry , M. L. Kapur , P. K. and Templeton , J. G. C. (1991) avoids spectral decomposition and well suited method for small and large matrices. Numerical techniques such as Ranga Kutta, Euler, Taylor and Randomization have been used to find transient solutions. Whereas first three have been employed in solving differential equation, the later one is particular suited for solving queueing problems. In

order to get greater accuracy one need to increase the number of steps. This together the number of simultaneous equations to be solved shows down the solutions process considerably Recently Sharma and Das (1988) have obtained transient solutions to a special category of Markovian model using eigen values of matrices in queueing theory. Standard numerical packages are used to obtained values, which are difficult to obtained when the matrices are large and a results computational difficulties arises.

Chaudhry, M.L., Kapur , P.K. and Templeton, J.G.C. (1991) have made attempt to obtained similar results in discrete time far finite waiting space problems in queueing theory. Since the transient solution depend on the initial state of system, it is intersecting to know the effect on the system behaviour. Kabayashi (1988) has discussed the several system which operate at discrete time many machine cycling of a processar and several other examples in computer science.

Chaudhry, M.L., Kapur. P.K. and Templeton, J.G.C. (1991) give the transient solutions for a general class discrete time models in queueing theory. In this they have assumed that the queue consists of finite waiting space. Interarrival and service probabilities are dependent on the state of the system, the inter arrival and service probabilities are dependent on the state of the system, the inter arrival and service time distributions are geometric but independent of time and queue discipline is first in first out.

By using these assumptions firstly they make the difference equations. To salve these equations they use the method of probability generating functions write these equations in matrix form. Cramer rule has been applied for finding the solutions of these equations. Explicit closed form expressions for distributions has been obtained in terms of the roots of a characteristic equation.

To find the eigen values or characteristics roots they make use of QROOT software package which is developed at Royal Military College at Canada by M.L. Chaudhry (1992).

For the analysis of the model the following notations are used

X_k – Number of customers in queue at epoch k.

N – Size of waiting space.

λ_n – Inter arrival probability when n customers are in the system

μ_n – Service probability when n customers are in the system.

$\phi_n = \mu_n (1 - \lambda_n)$

$\psi_n = \lambda_n (1 - \mu_n)$

Here P_m (n) denote the probability that the system is in the n^{th} state at the beginning of m^{th} epoch X_k, $k \geq 0$ in an interger valued discrete stochastic process taking value (0,1,2.....,N). X_k=n ($0 \leq n \leq N$) implies that there are n customers in the system at epoch k. The difference equations are

$$P_{m+1}(0) - P_m(0) = -\psi_0 P_m(0) + \phi_1 P_m(1) \qquad \text{..........(1.1)}$$

$$P_{m+1}(n) - P_m(n) = -P_m(n)\ (\phi_n + \psi_n) + P_m(n-1)\,\psi_{n-1} + P_m(n+1)\,\phi_{n+1}$$

$$; 1 \leq n \leq (N-1) \qquad \text{..........(1.2)}$$

$$P_{m+1}(N) - P_m(N) = -\phi_N P_m(N) + P_m(N-1)\,\psi_{N-1} \qquad \text{........ (1.3)}$$

and $P_0(i) = i \qquad ; 0 \leq i \leq N$

Let P_z (n) be the p.g.f. of P_m (n) defined as

$$P_z(n) = \sum_{m=0}^{\infty} z^m P_m(n) \qquad |z| \leq 1$$

Now taking the p.g.f. of equation (1.1), (1.2) and (1.3), we have

$$P_{m+1}(0)\, z^m - P_m(0)\, z^m = -\psi_0 P_m(0)\, z^m + \phi_1 P_m(1) z^m.$$

$$\sum_{m=0}^{\infty} P_{m+1}(0)\, z^m - \sum_{m=0}^{\infty} P_m(0)\, z^m = -\psi_0 \sum_{m=0}^{\infty} P_m(0)\, z^m + \phi_1 \sum_{m=0}^{\infty} P_m(1) z^m.$$

$$\left[\frac{1}{z}\right](P_z(0) - P_0(0)) - P_z(0) = -\psi_0 P_z(0) + \phi_1 P_z(1)$$

$$\left[\frac{1-z}{z}\right] P_z(0) - \frac{1}{z} P_0(0) = -\psi_0 P_z(0) + \phi_1 P_z(1)$$

$$(s+\psi_0)\, P_z(0) - \phi\, P_z(1) = \frac{1}{z} P_0(0) \qquad \text{..........(1.4)}$$

Wher $s = \dfrac{1-z}{z}$

$$\sum_{m=0}^{\infty} P_{m+1}(n)\, z^m - \sum_{m=0}^{\infty} P_m(n)\, z^m = \sum_{m=0}^{\infty} P_m(n)\, z^m (-\phi_n - \psi_n) + \psi_{n-1} \sum_{m=0}^{\infty} P_m(n-1) z^m.$$

$$+ \phi_{n+1} \sum_{m=0}^{\infty} P_m(n-1)\, z^m$$

$$\frac{1}{z}\{P_z(n) - P_0(n)\} - P_z(n) = (-\phi_n - \psi_n)\, P_z(n) + \psi_{n-1} P_z(n-1) + \phi_{n+1} P_z(n+1)$$

$$\left(\frac{1-z}{z}\right) P_z(n) - \frac{1}{z} P_0(n) = (-\phi_n - \psi_n)\, P_z(n) + \psi_{n-1} P_z(n-1) + \phi_{n+1} P_z(n+1)$$

$$(s + \phi_n + \psi_n)\, P_z(n) - \psi_{n-1} P_z(n-1) - \phi_{n+1} P_z(n+1) = (1/z)\, P_0(n)$$

$$-\psi_{n-1} P_z(n-1) + (s + \phi_n + \psi_n)\, P_z(n) - \phi_{n+1} P_z(n+1) = (1/z)\, P_0(n)$$

$$;\ 1 \le n \le N-1 \qquad \text{..........(1.5)}$$

$$\sum_{m=0}^{\infty} P_{m+1}(N)\, z^m - \sum_{m=0}^{\infty} P_m(N)\, z^m = -\phi_N \sum_{m=0}^{\infty} P_m(N)\, z^m + \psi_{N-1} \sum_{m=0}^{\infty} P_m(N-1) z^m.$$

$$(1/z)\{P_z(N) - P_0(N)\} - P_z(N) = -\phi_N P_z(N) + \Psi_{N-1} P_z(N-1)$$

$$\left(\frac{1-z}{z}\right) P_z(N) - (1/z)\,P_0(N) = -\,\phi_N P_z(N) + \psi_{N-1} P_z(N-1)$$

$$(s+\phi_N)\,P_z(N) - \psi_{N-1} P_z(N-1) = 1/z\,P_0(N)$$

$$-\,\psi_{N-1}\,P_z(N-1) + (s+\phi_N)\,P_z(N) = (1/z)\,P_0(N)$$

$$\text{Where } s = \frac{1-z}{z} \qquad \text{............(1.6)}$$

Now we have

$$\text{A.P.} = [\delta_{k0}, \delta_{k1}, \text{-----------------}, \delta_{kN}]' \qquad \text{............(1.7)}$$

Where A is real tridiagonal (N+1) × (N+1) Matrix, p is column vector and δ_{ki} is a kronecker delta defined as

$$\delta_{ki} = \begin{cases} 1/z\ ; & k = i \\ 0\ ; & \text{otherwise} \end{cases}$$

Assuming $\mu_0 = 0$ and $\lambda_N = 0$

$$A(s) = \begin{array}{c|ccccccccc|} & 0 & 1 & 2 & - & - & - & N-2 & N-1 & N \\ \hline 0 & s+\psi_0 & -\phi_1 & 0 & - & - & - & 0 & 0 & 0 \\ 1 & -\psi_0 & s+\phi_1+\psi_1 & -\phi_2 & - & - & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & -- & & - & - & - \\ - & - & - & - & - & - & - & - & - & - \\ N-1 & 0 & 0 & 0 & - & - & - & -\psi_{N-2} & s+\phi_{N-1}+\psi_{N-1} & -\phi_N \\ N & 0 & 0 & 0 & - & - & - & - & -\psi_{N-1} & s+\phi_N \end{array}$$

and

.........(1.8)

$$P = \begin{bmatrix} P_z(0) \\ P_z(1) \\ - \\ - \\ P_z(N) \end{bmatrix}_{(N+1) \times 1} \qquad \ldots\ldots(1.9)$$

From (1.8) Cramer's Rule $P_z(n)$ are explicitly determined as

$$P_z(n) = \frac{|A_{n+1}(s)|}{|A(s)|} \quad ; |z| \leq 1 \qquad \ldots\ldots(1.10)$$

Applying some row and column transformation on $|A(s)|$ it may be expressed as $s|D(s)|$, where D(s) is the matrix of order N×N given by

$$\begin{array}{cc} & \begin{array}{ccccccc} 1 & 2 & 3 & - - - & N-2 & N-1 & N \end{array} \\ \begin{array}{c} 1 \\ 2 \\ - \\ - \\ - \\ N-1 \\ N \end{array} & \begin{bmatrix} s+\phi_1+\psi_0 & -\sqrt{(\phi_1\,\psi_1)} & 0 & - - - & - & - & - \\ -\sqrt{(\phi_1\,\psi_1)} & s+\phi_2+\psi_1 & -\sqrt{(\phi_2\,\psi_2)} & - - - & - & - & - \\ - & - & - & - - - & - & - & - \\ - & - & - & - - & - & - & - \\ - & - & - & - - - & - & - & - \\ & & & - - - & -\sqrt{(\phi_{N-2}\,\psi_{N-2})} & s+\phi_{N-2}+\psi_{N-2} & \sqrt{(\phi_{N-1}\,\psi_{N-1})} \\ & & & - - - & 0 & -\sqrt{(\phi_{N-1}\,\psi_{N-1})} & s+\phi_N+\psi_{N-1} \end{bmatrix} \end{array}$$

If D(s) is expanded it will be a polynomial of degree N. The roots of the polynomial $|A(s)|$ are real, negative and distinct (one root being zero). Let α_k (k = 0,1,2…..N) be the root of $|A(s)|$ with

$\alpha_0 = 0$ then

$$|A(s)| = s \prod_{k=1}^{N} (s - \alpha_k)$$

and hence

$$P_z(n) = \frac{|A_{n+1}(s)|}{s \prod_{k=1}^{N} (s - \alpha_k)} \qquad 0 \le n \le N \qquad \text{......(1.11)}$$

resolving the right hand side of $P_z(n)$ into partial fraction replacing s by (1-z)/z using initial condition and comparing coefficient of z^m, we have

$$P_m(n) = b_n + \begin{cases} \prod_{r=n}^{i-1} \phi_{r+1} \sum_{k=1}^{N} \alpha_{kr} (1+\alpha_k)^m & 0 \le n < i \\ \sum_{k=1}^{N} \alpha_{kn} (1+\alpha_k)^m & n = i \\ \prod_{r=i}^{n-1} \psi_r \sum_{k=1}^{N} \alpha_{kn} (1+\alpha_k)^m & i < n \le N \end{cases}$$

$$\text{...............(1.12)}$$

where $\alpha_{kn's}$ and $b_{n's}$ are defined as

$$\alpha_{kn} = \begin{cases} \dfrac{C_{N-i}(\alpha_k) D_n(\alpha_k)}{\alpha_k \prod_{j=1}^{N} (\alpha_k - \alpha_j)} & 0 \le n < i \\ \dfrac{C_{N-i}(\alpha_k) D_i(\alpha_k)}{\alpha_k \prod_{j=1}^{N} (\alpha_k - \alpha_j)} & n = i \\ \dfrac{C_{N-n}(\alpha_k) Di(\alpha_k)}{\alpha_k \prod_{j=1, j \ne k}^{N} (\alpha_k - \alpha_j)} & i < n \le N \end{cases} \qquad \text{..........(1.13)}$$

and

$$b_n = \frac{C_{N-n}(0)\, D_n(0)}{\prod\limits_{k=1}^{N} (-\alpha_k)} \qquad 0 \leq n \leq N \qquad (1.14)$$

where $C_n(s)$ and $D_n(s)$ being the determinants obtained by the bottom right and top left and ($n \times n$) square matrices formed from $A(s)$ such that

$$|A(s)| = C_{N+1}(s) = D_{N+1}(s)$$

for convenince, we write $C_n(s)$ and $D_n(s)$ as C_n and D_n respectively then,

$$C_i = [\, s + \phi_{N+1-i} + \psi_{N+1-i}\,]\, C_{i-1} - [\phi_{N+2-i}\, \psi_{N+1-i}]\, C_{i-2}, \qquad 2 \leq i \leq N+1 \qquad \ldots\ldots(1.15)$$

And

$$D_i = [\, s + \phi_{i-1} + \psi_{i-1}]\, D_{i-1} - [\phi_{i-1} \psi_{i-2}]\, D_{i-2} \qquad 2 \leq i \leq N+1 \qquad \ldots\ldots(1.16)$$

Using QROOT software packages, we find the root α_k called the characteristic equation of $A(s)$ after finding the roots they discuss many cases and find the numerical results.

1.4- DISCRETE TIME TRANSIENT SOLUTION FOR GEOM (n) /GEOM (n) /2 /N WITH HETEROGENEOUS SERVER

We assume that the inter arrival probabilities and service time probabilities of first and second server to be geometrically distributed with parameter λ, μ_1, and μ_2 respectively. We also assume that $\mu_1<\mu_2$ that is the service time probability for first server is less than that of second server. we are considering modified queue discipline i.e. the first arriving unit from amongst the initial number of unit present of the start of the service join the first counter for service.

Therefore the arriving unit goes to the counter which it find free, the maximum number of customer in the system is restricted to N. We further assume that there is no unit initial waiting at the time t=o when the service starts

Notation:-

X_k:- denote the number of customer at epoch k

N:- Size of waiting space.

λ:- interarrival probability of a customer

μ_1:- service probability of a customer for server one

μ_2:- service probability of a customer for server two

$\phi_1= \mu_1(1-\lambda)$, $\phi_2 = \mu_2(1-\lambda)$, $\Psi=\lambda(1-\mu_1-\mu_2)$

Analysis of the model :

Let $P_m(n)$ (n=0, 1, 2; --N) denote the probability that the system is in the n^{th} state at the beginning of the m^{th} epoch or time slot. Let X_k be the number of customers in the

system at discrete time epoch k. Then X_k, $k \geq 0$ is an integer valued discrete stochastic process taking values 0 ,1 ,----------N. $X_k = n$ $(0 \leq n \leq N)$ implies that there are n customers in the system at epoch k .

The following difference equations can be written as

$$P_{m+1}(0) - P_m(0) = -\lambda P_m(0) + \phi_1 P_m(1) \qquad \text{.......} (1.17)$$

$$P_{m+1}(1) - P_m(1) = -P_m(1)(\psi + \phi_1 + \phi_2) + \lambda P_m(0) + (\phi_1 + \phi_2) P_m(2) \qquad \text{.......} (1.18)$$

$$P_{m+1}(n) - P_m(n) = -P_m(n)(\psi + \phi_1 + \phi_2) + \psi P_m(n-1) + (\phi_1 + \phi_2) P_m(n+1) \qquad \text{.......}(1.19)$$

$$2 \leq n \leq N-2$$

$$P_{m+1}(N-1) - P_m(N-1) = -P_m(N-1)(\psi + \phi_1 + \phi_2) + \psi P_m(N-2) + (\mu_1 + \mu_2) P_m(N) \text{ ..}(1.20)$$

$$P_{m+1}(N) - P_m(N) = \psi P_m(N-1) - (\mu_1 + \mu_2) P_m(N) \qquad \text{.............}(1.21)$$

Where $P_0(i) = 1$; $0 \leq i \leq N$

Let P(n) be the steady state distribution i.e

$$\lim_{m \to \infty} P_m(n) = P(n)$$

Let $P_z(n)$ be the probability generating function (p.g.f.) of $P_m(n)$ defined as

$$G(z,n) = P_z(n) = \sum_{m=0}^{\infty} z^m P_m(n) \quad |z| \leq 1$$

Taking the p.g.f of equation (1.17) to (1.21) form this we multiply the equation by z^m and taking summation from 0 to ∞ for m and using $(1-z)/z = s$, we get

$$(s+\lambda) P_z(0) - \phi_1 P_z(1) = 1/z \qquad \text{.......}(1.22)$$

$$-\lambda P_z(o) + (s+\psi+\phi_1+\phi_2) P_z(1) - (\phi_1+\phi_2) P_z(2) = 1/z - \qquad \text{......}(1.23)$$

$$-\psi P_z(n-1) + (s+\psi+\phi_1+\phi_2) P_z(n) - (\phi_1+\phi_2) P_z(n+1) = 1/z \qquad \text{......}(1.24)$$

$$2 \leq n \leq N-2$$

$$-\psi P_z(N-1) + (s+\phi_1+\phi_2) P_z(N-1) - (\mu_1+\mu_2) P_z(N) = 1/z \qquad \ldots\ldots\ldots(1.25)$$

$$-\psi P_z(N-1) + (s+\mu_1+\mu_2) P_z(N) = 1/z \qquad \ldots\ldots(1.26)$$

These equation (1.22) to (1.26) can be written as the matrix form

$$AP = [\delta_{k0}, \delta_{k1}, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\delta_{kN}]' \qquad \ldots\ldots(1.27)$$

Where A is a real tridiagonal $(N+1) \times (N+1)$ matrix , P is a column vector of order $(N+1) \times 1$ and δ_{ki} is the Kronecker Delta defined as

$$\delta_{ki} = \begin{cases} 1/z \;; & k = i \\ 0 \;; & \text{otherwise} \end{cases}$$

We have

$$A(s) = \begin{array}{c} \\ 0 \\ 1 \\ - \\ - \\ - \\ - \\ N-2 \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 0 & 1 & 2 & - & - & - & N-2 & N-1 & N \end{array} \\ \begin{bmatrix} s+\lambda & -\phi_1 & 0 & - & - & - & 0 & 0 & 0 \\ -\lambda & s+\psi+\phi_1+\phi_2 & -(\phi_1+\phi_2) & - & - & - & 0 & 0 & 0 \\ 0 & -\psi & (s+\phi_1+\phi_2) & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & (s+\psi+\phi_1+\phi_2) & -(\phi_1+\phi_2) & 0 \\ 0 & 0 & 0 & 0 & - & & -\psi & s+\psi+\phi_1+\phi_2 & -(\mu_1+\mu_2) \\ 0 & 0 & 0 & - & - & - & 0 & -\psi & s+\mu_1+\mu_2 \end{bmatrix} \end{array}$$

And $P = [P_z(0), P_z(1), \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots P_z(N)]'$

From equation (1.27) , using Cramer's rule $P_z(n)$ are explicitly determined as

$$P_z(n) = \frac{|A_{n+1}(s)|}{|A(s)|} \qquad ; 0 \le n \le N$$

Where $A_{n+1}(s)$ is obtained from A(s) by replacing the $(n+1)^{th}$ column of A(s) by the right hand side of equation (1.27) and $|A(s)|$ is the determinate of A(s)

Applying some row and column transformation on $|A(s)|$, it may be expressed as $s|D(s)|$ is a real symmetric tridiagonal matrix of order (N×N).

$$
D(s)=\begin{bmatrix}
s+(\phi_1+\lambda) & -\sqrt{(\phi_1\psi)} & 0 & - - - & 0 & 0 & 0 \\
-\sqrt{(\phi_1\psi)} & s+(\phi_1+\phi_2)+\psi & -\sqrt{\psi(\phi_1+\phi_2)} & - - - & 0 & 0 & 0 \\
- & - & - & - - - & - & - & - \\
- & - & - & - - & - & - & - \\
- & - & - & - - - & - & - & - \\
0 & 0 & 0 & - - - & -\sqrt{\psi}(\phi_1+\phi_2) & s+\Psi+(\phi_1+\phi_2) & -\sqrt{\psi\phi_1} \\
0 & 0 & 0 & - - - & & -\sqrt{\psi\phi_1} & s+\Psi+(\mu_1+\mu_2)
\end{bmatrix}
$$

$|D(s)|$ is a polynomial of degree N of s. The roots of $|D(s)|$ are the negatives of the eigenvalues of the matrix D(0). Hence the roots of the polynomial $|A(s)|$ are real, negative and distinct . Let α_k (k = 0, 1,2,.....N) be the roots of $|A(s)|$ with $\alpha_0=0$ then,

$$|A(s)| = s\prod_{k=1}^{N}(s-\alpha_k)$$

and hence

$$P_z(n) = \frac{|A_{n+1}(s)|}{s\prod_{j=1}^{N}(s-\alpha_j)} \quad ; \qquad 0\le n\le N$$

Resolving the R.H.S of $P_z(n)$ in to partial fractions, replacing s by (1-z)/z using initial conditions and comparing coefficient of z^m, we can find out the value $P_m(n)$. It will be bounded for $|1+\alpha_k| < 1$.

This work in that sense gives impetus to the analysis of discrete time models with heterogeneous servers.

1.5 DISCRETE-TIME TRANSIENT SOLUTION FOR A FIRST PASSAGE TIME DISTRIBUTION IN QUEUEING THEORY

Most of the queueing theory literature concentrates on finding the steady state solution or approximations. Chaudhry M.L., Agarwal, M. and Templetion, J.G.C. (1992) have mostly concentrated on this using the technique of roots. Earlier attempts at finding transient solution can be attributed to Takacs, L (1962) and Morse P.M. (1958). However, there are computational difficulties with their methods.

Now with the increased skill available in computations with the use of computers, researchers especially in computer science have started looking for transient solutions and easy to compute closed form solutions. Recently, Sharma, O.P. and Das, S. (1988) have provided transient solutions to a class of Markovian models in Queueing Theory.

The transient solution is not independent of the initial state of the system it is interesting to know its effect on the system's behaviour. Some system may not exist long enough to reach their steady state.

There are several systems which operate at discrete times see Kobayashi H. (1983) As a result, it becomes important to study them.

A discrete time Queueing model for the first passage time distribution to a absorbing state given the initial state. We give closed form solution to this class of problems in terms of the root of a polynomial in z-transform and results are computed even when the matrix involved are large. It is also shown, how the results for the continuous case can be obtained.

Result presented in this section further unify the treatment given by Chaudhry M.L., Kapur P.K., Templeton J.G.C. (1991).

We assume that

1. The queue consists of finite waiting space.
2. Inter-arrival and service probabilities are dependent on the state of the system.
3. Inter-arrival and service time distributions are geometric but independent of time.
4. Queue discipline is first in first out.

The notations used are :

X_k	:	Number of customer in the system at epoch k.
N	:	Size of the queue (maximum)
λ_n	:	Inter-arrival probability when n customer are in the system.
μ_n	:	Service probabilities when n customer are in the system.
ψ_n	:	$n(1 - \mu_n)$
ϕ_n	:	$n(1 - \lambda_n)$
h	:	Absorbing barrier ($h \geq 0$) ($h<n$)

ANALYSIS OF THE MODEL :

Let X_k be the number of customers in the system at discrete-time epoch k. Then X_k, $k\geq 0$ is an integer-valued discrete stochastic process taking values $(h, h+1, \ldots\ldots\ldots N)$. $X_k = n$ ($h\leq n\leq N$) implies that there are n customers in the system at discrete time epoch k. When a customer arrives or leaves, a discontinuity in the stochastic process occurs. The process X_k behaves as a discrete-time Markov process and represents the state of the system.

Denote the probability that the system in the state n at the beginning of the m^{th} epoch as $P_m(n)$, $(n=h,h+1..........,N)$

Backward First Time-Distribution Analysis :

Consider the case for the elapsed time slot m, the following difference equations may be easily written before the queue length for the first time reaches h.

$$P_{m+1}(h) - P_m(h) = \phi_{h+1} P_m(h+1) \qquad ....(1.28)$$

$$P_{m+1}(h+1) - P_m(h+1) = -(\psi_{n+1}+\phi_{n+1}) P_m(h+1) + \phi_{h+2} P_m(h+2) \qquad ....(1.29)$$

$$P_{m+1}(n) - P_m(n) = -(\psi_n+\phi_n) P_m(n) + \psi_{n-1} P_m(n-1) + \phi_{n+1} P_m(n+1) \qquad ....(1.30)$$

$$(h+2) \le n \le (N-1)$$

$$P_{m+1}(N) - P_m(N) = -\phi_N P_m(N) + \psi_{N-1} P_m(N-1) \qquad ....(1.31)$$

Where $\lambda_N=0$ and $P_0(i) = 1$, $\qquad h \le i \le N$

Let P_n be the steady state distribution, i.e.

$$\lim_{m\to\infty} P_m(n) = P(n).$$

If such a distribution exists, it is unique. Solving (1.28) to (1.31) for the stationary case, we get

$$P(h) = 1, \qquad P(1) = 0, \qquad h+1 \le 1 \le N$$

Let $P_Z(n)$ be the probability generating function (p.g.f.) of $P_m(n)$ defined as

$$G(z,n) = P_z(n) = \sum_{m=0}^{\infty} z^m P_m(n), \qquad |z| \le 1$$

Taking the p.g.f. of equations (1.28) to (1.31), we have

$$AP = [\delta_{kh}, \delta_{k(h+1)}.............\delta_{kN}]' \qquad ......(1.32)$$

Where A is real tri-diagonal (N-h+1) x (N-h+1) matrix, P is a column vector and δ_{ki} is the kronecker delta defined as

$$\delta_{ki} = \begin{cases} 1/z & ; \quad k=i \\ 0 & ; \quad \text{otherwise} \end{cases}$$

Definning s= (1-z)/z, we have

$$A(s) = \begin{array}{c c}
 & \begin{array}{ccccccccc} h & h+1 & h+1 & h+3 & - & - & - & N-1 & N \end{array} \\
\begin{array}{c} h \\ h+1 \\ h+2 \\ . \\ \\ . \\ N-1 \\ N \end{array} &
\left[\begin{array}{ccccccccc}
s & -\phi_{h+1} & 0 & - & - & - & - & - & \\
0 & s+\psi_{h+1}+\phi_{h+1} & -\phi_{h+2} & - & - & - & - & - & - \\
. & -\psi_{h+1} & s+\psi_{h+2}+\phi_{h+2} & -\phi_{h+3} & - & - & - & & \\
- & - & - & - & - & - & - & - & - \\
 & & & & - & & & & \\
- & - & - & - & - & - & - & & \\
- & - & - & - & & & \psi_{N-2} & s+\psi_{N-1}+\phi_{N-1} & \\
- & - & - & - & - & - & & -\psi_{N-1} & s+\phi_N
\end{array}\right]_{(N-h+1) \times (N-h+1)}
\end{array}$$

$$P = \begin{bmatrix} P_z(h) \\ P_z(h+1) \\ - \\ - \\ P_z(N) \end{bmatrix}_{(N-h+1) \times 1}$$

From equation (1.32), using Cramer's Rule $P_z(n)$ are explicity determined as

$$P_z(n) = \frac{|A_{n-h+1}(s)|}{|A(s)|} \qquad h \le n \le N$$

Where A_{n-h+1} is obtained from A(s) by replacing the $(n-h+1)^{th}$ column of A(s) by R.H.S of (1.32) and $|A(s)|$ is the determinant of A(s).

Applying some row and column transformation on $|A(s)|$, it may be expressed as $S|D(s)|$,where D(s) is a real, symmetric, tri-diagonal matrix of order (N-h) x (N-h) specifically D(s) =

$$
\begin{bmatrix}
s+(\phi_{h+1}) & -\sqrt{(\phi_{h+1}\psi_{h+1})} & - & - - - - & - & 0 \\
-\sqrt{(\phi_{h+1}\psi_{h+1})} & s+(\phi_{h+1}+\psi_{h+1}) & - & - - - - & - & 0 \\
 & & \sqrt{(\phi_{h+2}\psi_{h+2})} & & & \\
- & - & - & - - - - & - & - \\
- & - & - & - - \quad - & - & - \\
- & - & - & - - - - & - & - \\
- & - & - & - - - \; -\sqrt{(\phi_{N-2}\psi_{N-2})} & s+(\phi_{N-1}+\psi_{N-1}) & -\sqrt{(\phi_{N-1}\psi_{N-1})} \\
- & - & - & - - - - & -\sqrt{(\phi_{N-1}\psi_{N-1})} & s+(\phi_{N}+\psi_{N-1})
\end{bmatrix}
$$

D(s) is a polynomial of degree (N-h) in s. It may be noted that the roots of $|D(s)|$ are the negatives of the eigen values of the matrix D(0).

It may be observed that D(0) is a positive definite, symmetric, tri-diagonal matrix. The roots of the polynomial $|A(s)|$ are real, negative and distinct (one root is zero).

Let α_k (k=0,1,2..........,N-h) be the roots of $|A(s)|$ with $\alpha_0 = 0$. Then.

$$|A(s)| = s \prod_{k=1}^{N-h} (s - \alpha_k)$$

And hence

$$P_z(n) = \frac{|A_{n-h+1}(s)|}{s \prod_{j=1}^{N-h} (s - \alpha_j)}; \qquad h \le n \le N$$

Resolving the R.H.S of $P_z(n)$ into partial fractions and replacing s by $(1 - z)/z$, using initial conditions and comparing the coefficients of z^m, we get.

$$P_m(n) = 1 + \prod_{r=h}^{i-1} \phi_{r+1} \sum_{k=1}^{N-h} a_{kh} (1+\alpha_k)^m$$

$$P_m(n) = \prod_{r=n}^{i-1} \phi_{r+1} \sum_{k=1}^{N-h} a_{kn} (1+\alpha_k)^m, \qquad h < n \leq i$$

$$P_m(n) = \sum_{k=1}^{N-h} a_{kn} (1+\alpha_k)^m, \qquad n = 1$$

$$P_m(n) = \prod_{r=i}^{n-1} \psi_{r+1} \sum_{k=1}^{N-h} a_{kn} (1+\alpha_k)^m, \qquad i < n \leq N$$

$$a_{kh} = \frac{C_{N-i}(\alpha_k)}{\alpha_k \prod_{j=1 \neq k}^{N-h} (\alpha_k - \alpha_j)}$$

$$a_{kn} = \frac{C_{N-i}(\alpha_k)\, D_{n-h}(\alpha_k)}{\alpha_k \prod_{j=1 \neq k}^{N-h} (\alpha_k - \alpha_j)}, \qquad h < n \leq 1$$

$$a_{kn} = \frac{C_{N-i}(\alpha_k)\, D_{n-h}(\alpha_k)}{\alpha_k \prod_{j=1 \neq k}^{N-h} (\alpha_k - \alpha_j)}, \qquad n = 1$$

$$a_{kn} = \frac{C_{N-n}(\alpha_k)\, D_{i-h}(\alpha_k)}{\alpha_k \prod_{j=1 \neq k}^{N-h} (\alpha_k - \alpha_j)},$$

With $C_n(s)$ and $D_n(s)$ being the determinants obtained by the bottom right and top left (nxn) square matrices formed from A(s) such that

$$|A(s)| = C_{N-h+1}(s) = D_{N-h+1}(s)$$

$C_n(s)$ and $D_n(s)$ may be determined by the following recurrence relations.

Assuming $C_0(s)$ and $D_0(s) = 1$, $C_1(s) = s + \phi_N$, $D_1(s) = s$ and $\lambda_N = \psi_n = \phi_n = 0$

$$C_i(s) = (s + \psi_{N+1-i} + \phi_{N+1-i})\, C_{i-1} - \psi_{N+2-i}\,\phi_{N+2-i}\, C_{i-2} \qquad ; 2 \leq i \leq N-h+1$$

$$D_i(s) = (s + \psi_{h+i-1} + \phi_{h+i-1})\, D_{i-1} - \psi_{h+i-1}\,\phi_{h+i-1}\, D_{i-2} \qquad ; 2 \leq i \leq N-h+1$$

Using the standard IMSL packages one can find the eigen values and hence the zeros of the polynomial $|A(s)|$. Further using the recurrence relations for $C_i(s)$ and $D_i(s)$ can easily be evaluated.

Since $(1+\alpha_k)^m \to 0$ as $m \to \infty$ the steady state distribution P(n) is given by

$P(h) = 1.$ $\qquad P(n) = 0 \qquad\qquad h+1 \leq n \leq N$

Important Performance Measures :

Using closed form expressions for $P_m(n)$, some important measures can be analytically and numberically derived :-

1. Expected number of customers in the system (for fixed i)

$$E(X_m) = \sum_{n=h}^{N} n\, P_m(n)$$

2. If Y_m denotes the number of customers present in the queue.

$$E(X_m) = \sum_{n=h+r}^{N} (n-r)\, P_m(n)$$

3. Probability that the system state is greater than a given number c is given by

$$(c \geq h) \qquad \sum_{n=c}^{N} P_m(n)$$

4. Relaxation time which is a measure of length of time.

$$RT = 1/\left(\min_{i=1}^{N}\right)(-\mathrm{Re}\,(\alpha_i))$$

If $m >> RT$, $P_m(n) = P(n)$ $\qquad \forall n$

Forward First Passage Time :

Next we consider the absorbing barrier on the maximum queue size, we may write the difference equations as

$$P_{m+1}(0) - P_m(0) = (\psi_0)\,P_m(0) + \phi_1\,P_m(1)$$

$$P_{m+1}(n) - P_m(n) = -(\psi_n + \phi_n)\,P_m(n) + \psi_{n-1}\,P_m(n-1) + \phi_{n+1}\,P_m(n+1) \qquad 1 < n \leq N-2$$

$$P_{m+1}(N-1) - P_m(N-1) = -(\psi_{N-1} + \phi_{N-1})\,P_m(N-1) + \psi_{N-2}\,P_m(N-2)\,P_{m+1}(N)$$

$$P_{m+1}(N) - P_m(N) = -\psi_{N-1}\,P_m(N-1)$$

Where $\mu_0 = 0$ and $P_0(i) = 1$, $\qquad ; 0 < i \leq N$

Proceeding as above the steady state probabilities may be given as

$P(i) = 0 \qquad ; 0 < i \leq N-1 \qquad P(N) = 1$

The probabilities $P_m(n)$ may be expressed as

$$P_m(n) = \prod_{r=n}^{i-1} \phi_{r+1} \sum_{k=1}^{N} a_{kn}\,(1+\alpha_k)^m \qquad ; 0 < n \leq i$$

$$P_m(n) = \sum_{k=1}^{N} a_{kn}\,(1+\alpha_k)^m, \qquad ; n = i$$

$$P_m(n) = \prod_{r=i}^{n-1} \psi_r \sum_{k=1}^{N} a_{kn} (1+\alpha_k)^m, \qquad ; i < n \leq N$$

$$P_m(N) = 1 + \prod_{r=i}^{N-1} \psi_r \sum_{k=1}^{N} a_{kN} (1+\alpha_k)^m$$

$$a_{kn} = \frac{C_{N-i}(\alpha_k)\, D_n(\alpha_k)}{\alpha_k \prod_{j=1\neq k}^{N} (\alpha_k - \alpha_j)}, \qquad ; 0 < n \leq i$$

$$a_{kn} = \frac{C_{N-i}(\alpha_k)\, D_n(\alpha_k)}{\alpha_k \prod_{j=1\neq k}^{N} (\alpha_k - \alpha_j)}, \qquad ; n = i$$

$$a_{kn} = \frac{C_{N-n}(\alpha_k)\, D_i(\alpha_k)}{\alpha_k \prod_{j=1\neq k}^{N} (\alpha_k - \alpha_j)}, \qquad ; i < n \leq N$$

$$a_{kN} = \frac{D_i(\alpha_k)}{\alpha_k \prod_{j=1\neq k}^{N} (\alpha_k - \alpha_j)},$$

$C_n(s)$ and $D_n(s)$ are defined earlier.

i- CHANNEL BUSY PERIOD :

We define i-channel busy period ($o < i \leq N$) to begin with an arrival to the system at an epoch when there are (i-1) customers in the system to the very next epoch when there are again (i - 1) customers in the system. Assuming λ_n and μ_n to be the inter-arrival and service

probabilities respectively. When there are n customers in the system, the following difference equations may be written

$$P_{m+1}(i-1) - P_m(i-1) = \phi_i P_m(i)$$

$$P_{m+1}(i) - P_m(i) = -(\psi_i+\phi_i) P_m(i) + \phi_{i+1} P_m(i+1)$$

$$P_{m+1}(n) - P_m(n) = -(\psi_n+\phi_n) P_m(n) + \psi_{n-1} P_m(n-1) + \phi_{n+1} P_m(n+1) \quad i \le n \le N-1$$

$$P_{m+1}(N) - P_m(N) = -\phi_N P_m(N) + \psi_{N-1} P_m(N-1)$$

Where $\lambda_N = 0$ and $P_0(i) = 1$

The solution of these equations can be obtained as before with h=i-1 and h+1=i. $P_m(i-1)$ and $\phi_i P_m(i)$ are the probability distribution and probability mass function of the busy period.

NUMERICAL RESULTS :

The numerical results for only moderate values of N are given though were no problems even for large values of N.

Case (i) Backward First Passage Time :

Discrete Case : Assume r = 6, N=20, λ=0.8, μ=0.15, m=10, i = 4,5,6,........,20, h=4 ($h \le i \le N$) and $P_0(i)$ = 1/17. Table 1.51 gives the probabilities $P_m(n)$ for different i, the unconditional probabilities $Q_m(n)$ and steady state probability P(n). The last two rows give the values of $E(X_m)$ and $E(Y_m)$. For i=20, m=700 which is >>RT=47.

Continuous Case : Assume r=6, N=20 λ=0.8, μ=0.15, t=10, i=4,5,6,......,20. h=4 ($h \le i \le N$) and $P_0(i)$ = 1/17. Table 1.52 gives the probabilities $P_n(t)$ for different i, $Q_n(t)$ and P(n). For i=20 the time t=900 which is >>RT = 86.

Case (ii) Forward First Passage Time :

Discrete Case : Assume $r = 6$, $N=16$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=0,1,2,\ldots\ldots,16$, and $P_0(i) = 1/17$. Table 1.53 gives the probabilities $P_m(n)$ for different i,$Q_m(n)$ and $P(n)$.

Continuous Case : Assume $r=6$, $N=16$ $\lambda=0.8$, $\mu=0.15$, $t=10$, $i=0,1,2\ldots\ldots,16$.

$P_0(i) = 1/17$. Table 1.54 gives the probabilities $P_n(t)$ for different i, $Q_n(t)$ and $P(n)$.

Case (iii) i-channel busy period : Assume $r = 6$, $N=20$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=5,6$. Table 1.55 gives the probabilities $P_m(n)$ and the steady state probabilities.

Table 1.51 : Probabilities $P_m(n)$ P(n) and means for Geom(n)/Geom(n)/r/N with r = 6 N= 20, λ= 0.8 μ=0.15 m=10 i=4,5,6..............20, h =4,(h ≤i ≤n) and $Q_m(n)$ with $P_0(i)$ =1/17

i/n	4	5	6	18	19	20	$Q_m(n)$	P(n)
4	1.0000	0.5507	0.3163	0.0000	0.0000	0.0000	0.1217	1.0000
5	0.0000	0.1367	0.1874	0.0000	0.0000	0.0000	0.0370	0.0000
6	0.0000	0.2083	0.3022	0.0000	0.0000	0.0000	0.0717	0.0000
7	0.0000	0.0789	0.1376	0.0000	0.0000	0.0000	0.0627	0.0000
8	0.0000	0.0204	0.0444	0.0000	0.0000	0.0000	0.0597	0.0000
9	0.0000	0.0036	0.0102	0.0000	0.0000	0.0000	0.0590	0.0000
10	0.0000	0.0005	0.0017	0.0000	0.0000	0.0000	0.0589	0.0000
11	0.0000	0.0000	0.0002	0.0003	0.0000	0.0000	0.0588	0.0000
12	0.0000	0.0000	0.0000	0.0022	0.0003	0.0000	0.0588	0.0000
13	0.0000	0.0000	0.0000	0.0115	0.0022	0.0003	0.0588	0.0000
14	0.0000	0.0000	0.0000	0.0421	0.0115	0.0023	0.0588	0.0000
15	0.0000	0.0000	0.0000	0.1098	0.0421	0.0124	0.0587	0.0000
16	0.0000	0.0000	0.0000	0.2020	0.1103	0.0466	0.0583	0.0000
17	0.0000	0.0000	0.0000	0.2551	0.2039	0.1254	0.0566	0.0000
18	0.0000	0.0000	0.0000	0.2134	0.2619	0.2390	0.0517	0.0000
19	0.0000	0.0000	0.0000	0.1164	0.2290	0.3124	0.0417	0.0000
20	0.0000	0.0000	0.0000	0.0472	0.1388	0.2616	0.0217	0.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
E(xm)	4.0000	4.8954	5.4449	16.9809	17.8967	18.5685	11.1196	4.0000
E(Ym)	0.0000	0.1335	0.2648	10.9809	11.9867	12.5685	5.3999	0.0000

Table 1.52 : probabilities $P_n(t)$ .P(n) and means for M/M/R?N with $r = 6$, $N = 20, \lambda = 0.8$,

$\mu = 0.15$, $t = 10$, $i = 4,5,6,\ldots\ldots\ldots\ldots.20$. $h = 4$ ($h \leq i \leq n$) and $Q_n(t)$ with $p_o(i) = 1/17$,

i/n	4	5	6	18	19	20	$Q_n(t)$	P(n)
4	1.0000	0.8219	0.6693	0.0018	0.0009	0.0005	0.2459	1.0000
5	0.0000	0.0153	0.0269	0.0013	0.0007	0.0005	0.0169	0.0000
6	0.0000	0.0239	0.0422	0.0029	0.0016	0.0011	0.0285	0.0000
7	0.0000	0.0279	0.0489	0.0053	0.0032	0.0021	0.377	0.0000
8	0.0000	0.0277	0.0502	0.0092	0.0058	0.0041	0.0448	0.0000
9	0.0000	0.0244	0.0452	0.0151	0.0101	0..75	0.0498	0.0000
10	0.0000	0.0179	0.0371	0.0235	0.0167	0.0131	0.0532	0.0000
11	0.0000	0.0145	0.0282	0.0349	0.0262	0.0216	0.0553	0.0000
12	0.0000	0.0100	0.0199	0.0490	0.0392	0.0338	0.0564	0.0000
13	0.0000	0.0064	0.0131	0.0654	0.0557	0.0501	0.0568	0.0000
14	0.0000	0.0038	0.0081	0.0828	0.0750	0.0702	0.0565	0.0000
15	0.0000	0.0022	0.0048	0.1995	0.0957	0.0929	0.0556	0.0000
16	0.0000	0.0012	0.0026	0.1137	0.1155	0.1159	0.0541	0.0000
17	0.0000	0.0006	0.0014	0.1236	0.1317	0.1358	0.0520	0.0000
18	0.0000	00003	0.0007	0.1279	0.1416	0.1494	0.0492	0.0000
19	0.0000	0.0001	0.0003	0.1260	0.1437	0.1538	0.0457	0.0000
20	0.0000	0.0001	0.0002	0.1181	0.1367	0.1476	0.0416	0.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
$E(X_t)$	4.0000	4.8287	5.5862	15.9648	16.4125	16.6546	10.8242	4.0000
$E(Y_t)$	00000	0.4879	0.9517	9.9698	10.4150	10.6561	5.3330	0.0000

Table 1.53 :probabilities $p_m(n)$ and means for Geom/Geom(n)/r/N with $r=6$ $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=1,2,\ldots\ldots\ldots\ldots16$, and $Q_m(n)$ with $p_o(i)=1/17$.

i/n	0	1	2	14	15	16	$Q_m(n)$	P(n)
0	0.0004	0.0003	0.0002	0.0000	0.0000	0.0000	0.0001	0.0000
1	0.0075	0.0015	0.0033	0.0000	0.0000	0.0000	0.0012	0.0000
2	0.0510	0.0377	0.0269	0.0000	0.0000	0.0000	0.0093	0.0000
3	0.1763	0.1440	0.1144	0.0000	0.0000	0.0000	0.0399	0.0000
4	0.3255	0.3006	0.2707	0.0000	0.0000	0.0000	0.1014	0.0000
5	0.3047	0.3290	0.0.3448	0.0000	0.0000	0.0000	0.1567	0.0000
6	0.1219	0.1611	0.2034	0.0000	0.0000	0.0000	0.0427	0.0000
7	0.0120	0.0204	0.0324	0.0003	0.0000	0.0000	0.0753	0.0000
8	0.0007	0.0017	0.0036	0.0022	0.0003	0.0000	0.0614	0.0000
9	0.0000	0.0001	0.0003	0.0115	0.0022	0.0000	0.0591	0.0000
10	0.0000	0.0000	0.0000	0.0421	0.0144	0.0000	0.0586	0.0000
11	0.0000	0.0000	0.0000	0.1097	0.0411	0.0000	0.0579	0.0000
12	0.0000	0.0000	0.0000	0.2015	0.1046	0.0000	0.0552	0.0000
13	0.0000	0.0000	0.0000	0.2527	0.1832	0.0000	0.0478	0.0000
14	0.0000	0.0000	0.0000	0.2042	0.2062	0.0000	0.0338	0.0000
15	0.0000	0.0000	0.0000	0.1916	2.1228	0.0000	0.0154	0.0000
16	0.0000	0.0000	0.0000	0.0842	0.3282	1.0000	0.0842	1.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	4.2848	4.4840	4.6854	13.0336	14.2047	16.0000	8.2938	16.0000
E(Ym)	0.1481	0.2075	0.2799	8.0336	9.2047	11.0000	3.5080	11.0000

Table 1.54 : Probabilities $P_n(t)$ $P(n)$ and means for m/m /r/n with $r=6$ $\lambda=0.08$ $\mu=0.15$ $t=10$, $i = 0,1,2,\ldots\ldots16$, and $Q_n(t)$ with $p_o(i) = 1/17$

i/n	0	1	2	14	15	16	$Q_n(t)$	P(n)
0	0.0159	0.0123	0.0095	0.0000	0.0000	0.0000	0.0038	0.0000
1	0.0657	0.0545	0.0449	0.0003	0.0001	0.0000	0.0184	0.0000
2	0.1358	0.1199	0.1048	0.0015	0.0006	0.0000	0.0454	0.0000
3	0.1869	0.1747	0.1613	0.0043	0.0019	0.0000	0.0757	0.0000
4	0.1920	0.1893	0.1839	0.0019	0.0042	0.0000	0.0958	0.0000
5	0.1562	0.1619	0.1650	0.0153	0.0072	0.0000	0.0982	0.0000
6	0.1034	0.1123	0.1199	0.0212	0.0104	0.0000	0.0849	0.0000
7	0.0643	0.0735	0.0824	0.0294	0.0148	0.0000	0.0741	0.0000
8	0.0377	0.0454	0.0537	0.0389	0.0201	0.0000	0.0653	0.0000
9	0.0209	0.0266	0.0332	0.0484	0.0257	0.0000	0.0578	0.0000
10	0.0110	0.0147	0.0194	0.0561	0.0305	0.0000	0.0507	0.0000
11	0.0055	0.0078	0.0108	0.0600	0.0333	0.0000	0.0435	0.0000
12	0.0026	0.0039	0.0057	0.0583	0.0329	0.0000	0.0356	0.0000
13	0.0012	0.0018	0.0029	0.0502	0.0287	0.0000	0.0271	0.0000
14	0.0005	0.0008	0.0013	0.0364	0.021	0.0000	0.0180	0.0000
15	0.0002	0.0002	0.0006	0.0186	0.0108	0.0000	0.0088	0.0000
16	0.0002	0.0004	0.0007	0.5520	0.7578	1.0000	0.1969	1.0000
Sum	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xt)	4.2378	4.5005	4.7793	13.4019	14.6366	16.0000	8.5799	16.0000
E(Yt)	0.5530	0.6783	0.8278	8.4257	9.4257	11.0000	4.0560	11.0000

Table 1.55 : Probabilities $P_m(n)$ p(n) for geom. (n) /geom.(n) /r/n for i –channel busy period with $r=6$, $N=20$, $\lambda=0.8$, $\mu=0.15, m=10$, $i=5,6$.

i =5

n	$P_m(n)$	P(n)
4	0.5507	1.0000
5	0.1367	0.0000
6	0.2083	0.0000
7	0.0798	0.0000
8	0.0204	0.0000
9	0.0036	0.0000
10	0.0005	0.0000
11	0.0000	0.0000
12	0.0000	0.0000
13	0.0000	0.0000
14	0.0000	0.0000
15	0.0000	0.0000
16	0.0000	0.0000
17	0.0000	0.0000
18	0.0000	0.0000
19	0.0000	0.0000
20	0.0000	0.0000

i = 6

n	$P_m(n)$	P(n)
5	0.8524	1.0000
6	0.0108	0.0000
7	0.0186	0.0000
8	0.0227	0.0000
9	0.0231	0.0000
10	0.0208	0.0000
11	0.0169	0.0000
12	0.0127	0.0000
13	0.0088	0.0000
14	0.0057	0.0000
15	0.0035	0.0000
16	0.0020	0.0000
17	0.0010	0.0000
18	0.0005	0.0000
19	0.0003	0.0000
20	0.0002	0.0000

1.6 NUMERICAL COMPUTATIONS OF DISCRETE-TIME SOLUTION FOR A MULTI-SERVER QUEUE WITH BALKING AND RENEGING

We analyze a discrete time multi-server queue with balking and reneging given the initial state. We also discuss the case when the initial state is arbitrary. We give closed form solution to this class of problems in terms of roots of polynomial in z-transform and results are computed even when the matrices involved are large. Its is also shown, how the results in the continuous case can be obtained.

This section should simulate the study of discrete time models in other areas such as computer science. Finally extensive numerial computations were performed in order to judge the accuracy of the results. Case of Machine interference problems is also given.

Assumption :

1. The queue size is finite.
2. Inter-arrival and service probabilities are dependent on the state of the system.
3. Inter-arrival and service time distributions are geometric but independent of time.
4. Queue discipline is first come first serve.

Notations :

X_k : Number of customer in the queue at time epoch k.

N : Maximum queue size.

$\in_n$: Inter-arrival probability when n customer are in the system.

T_n : $\in_n (1 - \mu_n)$ $\phi_n : \mu_n (1 - \in_n)$

Analysis :

We develop a general discrete time Markov Model for a finite waiting space queueing system and analyze the effects of customer impatience on its transient behaviour. Impatience can be due to Balking, Reneging or both. Balking is the reluctance of a customer to join the queue upon arrival. Reneging is the reluctance of a customer to remain in the queue after joining it and leaving the queue without being serviced. The initial number of customers c will not renege because of their immediate entry to the service facility. Still these c customers join the queue with some balking probability.

We assume that inter-arrival and service times have geometric distributions with parameters $\in$ and μ. An arriving customer balking with probability n/N, n = 0,1,2,3...........N. Thus

inter-arrival probability may be defined as

$$\in_n = \in (1 - n/N) \qquad ; 0 \leq n \leq N$$

A customer may renege after joining the queue if hear she decides t certain waiting time will be larger that can be tolerated. This reneging time is assumed to have a geometric distribution with parameters Ω. Since any one of (n - c) customers may renege, the reneging probability may be expressed as.

$$\begin{array}{lll} 0 & \text{for} & 0 \leq n \leq c-1 \\ (n-c)\Omega & \text{for} & c \leq n \leq N \end{array}$$

Thus the service probability mat be expressed as

$$\mu_n = \begin{cases} n\mu & \text{for} \quad 0 \leq n \leq N \\ c\mu + (n-c)\Omega & \text{for} \quad n \leq c \leq N \end{cases}$$

Let X_k be the number of customers in the system at discrete time epoch k. Then X_k, $k \geq 0$ is an integer valued discrete stochastic process taking values 0,1,2,...........,N. $X_k = n$ $(0 \leq n \leq N)$ implies that there are n cutomers in the system. The process X_k behaves as a discrete-time Markov process and represent the state of the system.

Denote the probability that the system is in state n at the m[th] epoch as $P_m(n)$ $(0 \leq n \leq N)$. The following difference equations may easily be written.

$$P_{m+1}(0) = P_m(0)(1-\in) + P_m(1)\mu\ (1-(N-1)\in/n) \qquad \text{.............(1.33)}$$

$$P_{m+1}(n) = P_m(n)(1-(N-n)\in/n - n\mu + 2n\in\mu(N-n)/N) + Pm(n-1)\ (1-(n-1)\mu)(N-n+1)\in/N + P_m(n+1)\ (n+1)\mu\ (1-(N-n-1))\in/N)$$
$$1 \leq n \leq c-1 \qquad \text{.............(1.34)}$$

$$P_{m+1}(c) = P_m(c)(1-(N-c)\in/N - c\mu + 2c\in\mu\ (N-c)/N) + P_m(c-1)(1-(c-1)\mu)(N-c+1)\in/N + P_m(c+1)(c\mu+\Omega)\{1-(N-c-1)\in/N\} \qquad \text{..........(1.35)}$$

$$P_{m+1}(n) = P_m(n)(1-(N-n)\in/n - c\mu - (n-c)\Omega + 2(c\mu + (n-c)\Omega)(N-n)\in/N) + P_m(n-1)(1-c\mu-(n-c-1)\Omega)(N-n+1)\in/N + P_m(n+1)(c\mu+(n-c+1)\Omega)\{1-(N-n-1)\in/N\} \qquad c+1 \leq n \leq N-1 \qquad \text{.............(1.36)}$$

$$P_{m+1}(N) = P_m(N)(1-c\mu-(N-c)\Omega) + P_m(N-1)(1-c\mu-(N-1-1)\Omega)\in/N$$
$$\text{with } P_0(i) = 1, \quad 0 \leq i \leq N \qquad \text{.............(1.37)}$$

Let $P_z(n)$ be the p.g.f. of $P_m(n)$ defined as

$$P_z(n) = \sum_{m=0}^{\infty} z^m P_m(n) \qquad |z| \leq 1$$

Taking p.g.f. of equations (1.33) to (1.37) we get $AP = [\delta_{ko}, \delta_{k1} \text{..........} \delta_{kN}]'$ ....(1.38)

Where A is (N+1) x (N+1) ridiagonal matrix with real coefficient, p is a (N+1) x 1

column vector and δ_{k1} is the kronecker delta defined as

$$\delta_{ki} = \begin{cases} 1/z & ; \quad k=i \\ 0 & ; \quad \text{ofterwise} \end{cases}$$

Defining $s = (1 - Z)/z$ we have

$$A(s) = \begin{bmatrix} s+T_O & -\phi_1 & 0 & - & - & - & 0 & 0 & 0 \\ -T_O & s+\phi_1+T_1 & -\phi_2 & - & - & - & - & 0 & 0 \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & - & - & - & -T_{N-2} & s+T_{N-1}+\phi_{N-1} & -\phi_N \\ 0 & 0 & 0 & - & - & - & - & -T_{N-1} & s+\phi_N \end{bmatrix}$$

Where T_1 and ϕ_1 are defined above and

$$P = \begin{bmatrix} P_Z(0) \\ P_Z(n) \\ : \\ P_Z(N-1) \\ P_Z(N) \end{bmatrix}$$

From equation (1.38) using cramer rule, we may determine $P_Z(n)$ explicitely as

$$P_z(n) = \frac{|A_{n+1}(s)|}{|A(s)|} \qquad 0 \leq n \leq N$$

Where A_{n+1} (s) is obtained from A(s) by replacing the $(n+1)^{th}$ column of A(s) by the R.H.S. in (1.38) and $|A(s)|$ is the determinant of A(s).

We may observe that $|A(s)| = sg_N(s)$, Where $g_n(s)$ satisfies the recurrence relation.

$g_n(s) - (s+T_{N-1} + \phi_{N-n+1})\, g_{n-1}(s) + T_{N-n+1}\, \phi_{N-n+1}\, g_{n-2}(s) = 0, 1 \leq n \leq N$

$g_{-1}(s) = 0$ and $g_0(s) = 1$

$g_n(s)$ may also be expresed as the determinant & NxN real symmetric matrix g(s) as

$$g(s) = \begin{bmatrix} s+\phi_1+T_0 & \sqrt{\phi_1 T_1} & 0 & - & - & - \\ \sqrt{T_1\phi_1} & s+\phi_Z+T_1 & \sqrt{\phi_Z T_Z} & - & - & - \\ - & - & - & - & - & - \\ - & - & - & \sqrt{\phi_{N-Z} T_{N-Z}} & s+\phi_{N-1}+T_{N-1} & \sqrt{\phi_{N-1} T_{N-1}} \\ - & - & 0 & - & \sqrt{\phi_{N-1} T_{N-1}} & s+\phi_N+T_{n-1} \end{bmatrix}$$

The zeros of the $g_N(s)$ are the negatives of the eigen values of the matrix g(0). g(0) is a positive definite symmetric tri-diagonal matrix. Hence the eigen values are real, positive (>0) and distinct. Hence the root of $g_N(s)$ are real, negative and distinct. Let $\alpha_1, \alpha_2, \ldots\ldots\ldots\ldots \alpha_N$ be the roots of $g_N(s)$. Thus

$$P_Z(n) = \frac{|A_{n+1}(s)|}{s\prod_{i=1}^{N}(s-\alpha_1)} \qquad 0 \le n \le N$$

Resolving the R.H.S. of $P_Z(n)$ into partial fractions, replacing s by (1-z)/z and comparing the coeff. of z^m we have.

Case I :-

For $0 \le i \le c\text{-}1$

$$b_n + \frac{1!}{n!}\mu^{1-n} X_n \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m, \qquad 0 \le n \le i$$

$$P_m(n) = b_n + \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m, \qquad n = i$$

$$b_n + \frac{(n-1)!}{(N-n)!} (\in/N)^{n-1} \prod_{j=1}^{n-1} (1-j\mu) \sum a_{kn} (1-\alpha_k)^m, \qquad i \le n \le c$$

$$b_n + \frac{(n-1)!}{(N-n)!} (\in/N)^{n-1} \prod_{j=1}^{n-1} (1-j\mu) \prod_{k=c}^{n-1}(1-(c\mu + (k - c\Omega)) \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m, \qquad c \le n \le N$$

Where $X_n = \prod_{j=n}^{i-1} (1 - \in + (j+1)\in/N)$

Case II :-

For $c \le i \le N$

$$b_n + \frac{(c-1)!}{n!} \mu^{c-n-1} \prod_{k=1}^{i-c+1}\left(c\mu + (i\text{-}c\text{+}1\text{-}k)\Omega\right). X_n \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m, \qquad ; c \le n \le c\text{-}1$$

$$b_n + \sum_{k=1}^{i-n} \left(c\mu + (i\text{-}c\text{+}1\text{-}k)\Omega\right) X_n \sum_{k=1}^{n} a_{kn} (1-\alpha_k)^m, \qquad ; c \le n \le i$$

$$P_m(n) = b_n + \sum a_{Kn} (1-\alpha_K)^m, \qquad n = i$$

$$b_n + \frac{(n-1)!}{(N-n)!} (\in/N)^{n-1} \prod_{k=i}^{n-1}\left((i\text{-}c\mu\text{+}(k\text{-}c)\Omega\right) X_n \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m \qquad ; i\text{+}1 \le n \le N$$

Where a_{kn}'s are defined as.

$$a_{kn} = \begin{cases} \dfrac{C_{N-1}(\alpha_k)\, D_n(\alpha_k)}{\alpha_k \prod\limits_{j=1\neq k}^{n} (\alpha_k - \alpha_j)} & ; 0 \le n \le i \\[2ex] \dfrac{C_{N-n}(\alpha_k)\, D_1(\alpha_k)}{\alpha_k \prod\limits_{j=1\neq k}^{n} (\alpha_k - \alpha_j)} & ; i \le n \le N \end{cases}$$

With $C_n(s)$ and $D_n(s)$ being the determinants obtained by the bottom right and top left (nxn) square matrices farmed from A(s) such that

$$|A(s)| = C_{N+1}(s) = D_{N+1}(s)$$

$C_n(s)$ and $D_n(s)$ may be determined by the following recursive relations.

$$C_n(s) = (s+\phi_{N-n+1} + T_{N-n+1})\, C_{n-1}(s) - T_{N-n+1}\, \phi_{N-n+z}\, C_{n-Z}(s), \qquad 1 \le n \le N+1$$

With $C_0(s) = 1$ $C_{-1}(s) = 0$

$$D_n(s) = (s+\phi_{n-1} + T_{n-1})\, D_{n-1}(s) - \phi_{n-1}\, T_{n-z}\, D_{n-z}(s), 1 \le n \le N+1$$

With $D_0(s) = 1$ and $D_{-1}(s) = 0$

and

$$b_0^{-1} = \begin{cases} 1+\sum\limits_{C} {}^{N}C_r\, (\epsilon/N\mu)^r \dfrac{\prod\limits_{i=1}^{r}(1-(i-1)\mu)}{\prod\limits_{i=1}^{r}(1-\epsilon+i\epsilon/n)}, & C=N \\[3ex] 1+\sum\limits_{C}^{r=1} {}^{N}C_r\, (\epsilon/N\mu)^r \dfrac{\prod\limits_{i=1}^{r}(1-(i-1)\mu)}{\prod\limits_{i=1}^{r}(1-\epsilon+i\epsilon/n)} + \sum\limits_{r=c+1}^{N} \dfrac{N!\, \epsilon^r \prod\limits_{i=1}^{c}(1-(i-1)\mu) \prod\limits_{i=c+1}^{r}(1-(C\mu+(i-c-1)\Omega))}{(N-r)!C!\, \mu^C \mu^C \prod\limits_{i=c+1}^{r}(c\mu+(i-c)\Omega)(1-\epsilon-\epsilon i/N)} & ; c<N \end{cases}$$

$$\begin{bmatrix} {}^{N}C_r\, (\epsilon/N\mu)^r \dfrac{\prod\limits_{i=1}^{r}(1-(i-1)\mu)}{\prod\limits_{i=1}^{r}(1-\epsilon+i\epsilon/n)}, \\[3ex] \dfrac{N!\, \epsilon^r \prod\limits_{i=1}^{c}(1-(i-1)\mu) \prod\limits_{i=c+1}^{r}(1-(c\mu+(i-c-1)\Omega)}{N^r(N-r)!C!\, \mu^C \prod\limits_{i=c+1}^{r}(c\mu+(i-c)\Omega) \prod\limits_{i=1}^{c}(1-\epsilon+i\epsilon/N)} \quad ; c+1 \le r \le N \end{bmatrix}$$

It may be remarked that for probabilities $P_m(n)$ to remain bounded $|1+\alpha_k|<1$, which is true if $\phi_k + T_{k-1} < 1$. Under this condition, the sum of the absolute values of the elements in each row of the matrix D(0) is less than 2 and hence from the Gerschgorin's theorem $|\alpha_k| < 2$ (see) Hunter J.J. (1983), which implies $|1+\alpha_k| < 1$.

Using IMSL packages, we can get the eigen values (roots of g(0) and ($g_N(\delta)$).

Since $(1+\alpha_K)^m \rightarrow 0$ as $m\rightarrow\infty$, the steady state distribution of $P_m(n)$ may be defined as.

$$P(n) = \lim_{m\to\infty} P_m(n) = b_n \qquad 0 \le n \le N$$

It may be noted that teh values of $P_m(n)$ have been expressed as the sum of two expressions, one pertaining to the steady state and other to the transient state.

IMPORTANT PERFORMANCE MEASURE :

1. Mean number of customers in the system at epoch m

$$E(X_m) = \sum_{n=0}^{N} n\, P_m(n)$$

2. Mean number of customers in the queue at epoch m

$$E(Y_m) = \sum_{n=c}^{N} (n-c)\, P_m(n)$$

3. Probability there are r or more customer in the system at epoch m

$$\sum_{n=r}^{N} P_m(n)$$

4. Probability all servers are busy at epoch m

$$E(Z_m) = \sum_{n=c}^{N} P_m(n)$$

5. Relaxation Time (RT) (a meassure of the length of time required by the system to settle to its stready state). [Morse P.M. (1958)]. may be defined as.

$$RT = \frac{1}{\min\limits_{i=1}^{N} (-\alpha i)} \qquad \text{If } m>>RT \text{ then } Pm(n) \approx P(n)$$

6. The probability of balking at epoch m

$$E(A_m) = \sum_{n=0}^{N} (n/N)\, P_m(n)$$

7. The probability of waiting up to epoch m in the queue by those joining it.

$$E(B_m) = \sum_{n=c+1}^{N} (1-n)/N\, P_m(n)$$

GENERAL CASE :-

We have assumed that the initial queue size is fixed and equal to i i.e. 1/z occurs in only one position in the initial probability vector. This assumption is important when we are interested in the transient solution. The steady state solution does not depend on the initial probability vector.

When there are than one non-zero elements in the initial probability vector. The

probability $Q_m(n)$ (n=o,1,2...........N) defined as the probability of n customers in the system at epoch m irrespective of the state of the system may be defined as.

$$Q_m(n) = \sum_{i=0}^{N} P_m(n,i)\, P_0(i) \qquad\qquad 0 \le n \le N$$

Where $P_0(i)$ is the ith element of the initial probability vector and $P_m(n,i)$ is the probability of n sustomers in the system at epoch m assuming i as the initial number of customer.

CONTINUOUS TIME CASE :-

Leting $\in_n = \in_n \theta + 0(\theta)$, $\mu_n = \mu_n \theta + 0(\theta)$, m = t and m+1 = t + θ in equations (1.33) to (1.37), the difference equations in m can be transformaed into differential equations in t. The roots equation can be changed to get the solution. The roots equation can be changed to get the continuous time solution from the final discrete time solution. The roots of α_k of $|A(s)| = 0$ are transformed to $\alpha_k'\theta$. $(1 + \alpha_k'\theta)m$ tends to $e_k^{\,m}$ t in continuous time when t is divided into m sub-interval each of length θ such as t = mθ. The parameter s itself may be treated as the transform parameter in continuous time. R.H.S. of (1.38) will have 1 in the ith place instead of 1/z.

Treating Ω= 0, $\in_i = N\in$, (i = 0,1,2.........,N), b_1's represent the steady state probabilities for a geom(n)/geom(n)/c/N machine interence model.

NUMERICAL ANALYSIS :-

Case (I) Balking and Reneging :

Discrete Case :

Assume c = 5, N = 20, ∈ = 0.5, μ = 0.15, m = 10, Ω = 0.005, $\in_i = (1 - i/N)\in$, $\mu_i = \mu i$ for $0 \le i \le c$, $\mu_i = \mu c + (i - c)\Omega$ for $c \le i \le N$ and $P_0(i) = 1/21$. Table 1.61 gives the probabilities $P_m(n)$ for different i($0 \le i \le 20$), the unconditional probabilities $Q_m(n)$ and the steady state probabilities P(n). The last five rows give the values for $E(X_m)$, $E(Y_m)$, $E(Z_m)$, $E(A_m)$ and $E(B_m)$. The epoch to reach steady state is m ≈ 300 which is >> RT = 32.

Continuous Case :-

Assume c = 5, N = 20, ∈ = 0.5, μ = 0.15 t = 10, Ω = 0.005, $\in_i = (1 - i/N)\in$, $\mu_i = i\mu$ for $0 \le i \le c$ $\mu_i = c\mu + (i - c)\Omega$, for $c < i \le N$ and i($0 \le i \le 20$) , the unconditional probabilities $Q_t(n)$ and the steady state probabilities P(n). The last five rows give the values for $E(X_t)$, $E(Y_t)$, $E(A_t)$ and $E(B_t)$. The time to reach steady state is t ≈ 260 which is >>RT = 31.

Case (II) Machine Interference Model :-

Discrete Case :

Assumec=5, N=20, $\in$=0.04, μ=0.1, m=10, $\in_i$=(N-i)$\in$, μ_i=iμ for $0 \le i \le c$, μ_i=cμ for $c \le i \le N$ and $P_0(i) = 1/21$. Table 1.63 gives the probabilities $P_m(n)$ for $0 \le i \le 20$, $Q_m(n)$ and P(n). The epoch to reach steady state is m $\approx$220 which is >>RT=22.

Assume t=10 and rest of the parameters as in table 1.63, Table 1.64 gives the probabilities $P_t(n)$ for different i($0 \le i \le 20$), $Q_t(n)$ and P(n). The time reach to steady state is t $\approx$130 which is >>RT=20.

Finally the accuracy of the values (roots) that is needed in such problems is very very large because of the recurrence relations involved in computing the probabilties.

Table 1.61 probabilities $P_m(n)$, $Q_m(n)$, $P(n)$ and some important performance measures for Geom(n)/Geom(n)/c/N balking and reneging problem with C=5,N=20,m=10, ε=0.5, μ=0.05, Ω=0.005, i=0,1,2,………20 and $P_0(i)$= 1/21.

	0	1	2		18	19	20	$Q_m(n)$	P(n)
0	0.0060	0.0025	0.0010	…	0.0000	0.0000	0.0000	0.0005	0.0000
1	0.0480	0.0252	0.0124	…	0.0000	0.0000	0.0000	0.0045	0.0002
2	0.1552	0.1019	0.0620	…	0.0000	0.0000	0.0000	0.0183	0.0015
3	0.2673	0.2188	0.1643	…	0.0000	0.0000	0.0000	0.0432	0.0068
4	0.2712	0.2780	0.2574	…	0.0000	0.0000	0.0000	0.0674	0.0205
5	0.1673	0.2161	0.2468	…	0.0000	0.0000	0.0000	0.0754	0.0420
6	0.0654	0.1096	0.1580	…	0.0000	0.0000	0.0000	0.0721	0.0712
7	0.0167	0.0379	0.0711	…	0.0000	0.0000	0.0000	0.0679	0.1058
8	0.0026	0.0086	0.0219	…	0.0000	0.0000	0.0000	0.0654	0.1372
9	0.0003	0.0012	0.0045	…	0.0000	0.0000	0.0000	0.0645	0.1546
10	0.0000	0.0002	0.0006	…	0.0003	0.0000	0.0000	0.0642	0.1504
11	0.0000	0.0000	0.0000	…	0.0029	0.0005	0.0001	0.0642	0.1257
12	0.0000	0.0000	0.0000	…	0.0152	0.0040	0.0007	0.0641	0.0893
13	0.0000	0.0000	0.0000	…	0.0548	0.0203	0.0055	0.0631	0.0534
14	0.0000	0.0000	0.0000	…	0.1354	0.0690	0.0267	0.0637	0.0264
15	0.0000	0.0000	0.0000	…	0.2301	0.1603	0.0860	0.0619	0.0108
16	0.0000	0.0000	0.0000	…	0.2633	0.2525	0.1873	0.05 61	0.0034
17	0.0000	0.0000	0.0000	…	0.1939	0.2619	0.2725	0.0439	0.0008
18	0.0000	0.0000	0.0000	…	0.0843	0.1673	0.2533	0.0263	0.0001
19	0.0000	0.0000	0.0000	…	0.0184	0.0570	0.1359	0.0104	0.0000
20	0.0000	0.0000	0.0000	…	0.0014	0.0072	0.0320	0.0019	0.0000
Total	1.0000	1.0000	1.0000	…	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	3.6243	4.0819	4.5603	…	15.6806	16.4180	17.1554	9.9533	9.2255
E(Ym)	0.1077	0.2166	0.3869	…	10.6809	11.4180	12.1554	5.1826	4.2647
E(Zm)	0.2523	0.3736	0.5029	…	1.0000	1.0000	1.0000	0.8660	0.9711
E(Am)	0.1807	0.2041	0.2280	…	0.7840	0.8209	0.8578	0.4977	0.4613
E(Bm)	0.0584	0.1072	0.1727	…	0.2160	0.1791	0.1422	0.3339	0.4836

T able 1.62: Probabilities $P_t(n)$, $Q_t(n)$, $P(n)$ for m/m/c/n queue with t =10 and rest of the parameters as in table (1.61).

	0	1	2		18	19	20	$Q_m(n)$	P(n)
0	0.0209	0.0089	0.0038	...	0.0000	0.0000	0.0000	0.0017	0.0000
1	0.0891	0.0526	0.0286	...	0.0000	0.0000	0.0000	0.0095	0.0010
2	0.1807	0.1360	0.0932	...	0.0000	0.0000	0.0000	0.0256	0.0048
3	0.2314	0.2114	0.1767	...	0.0000	0.0000	0.0000	0.0463	0.0145
4	0.2087	0.2243	0.2219	...	0.0000	0.0000	0.0000	0.0625	0.0307
5	0.1391	0.1711	0.1954	...	0.0000	0.0000	0.0000	0.0667	0.0492
6	0.0763	0.1064	0.1390	...	0.0000	0.0000	0.0000	0.0673	0.0723
7	0.0349	0.05 46	0.0808	...	0.0000	0.0000	0.0000	0.0664	0.0974
8	0.0133	0.0232	0.0152	...	0.0002	0.0001	0.0000	0.0654	0.1194
9	0.0043	0.0083	0.0050	...	0.0008	0.0003	0.0001	0.0647	0.1327
10	0.0012	0.0025	0.0014	...	0.0029	0.0011	0.0004	0.0644	0.1327
11	0.0003	0.0006	0.0003	...	0.0095	0.0038	0.0004	0.0643	0.1186
12	0.0000	0.0001	0.0001	...	0.0265	0.0120	0.0049	0.0642	0.0935
13	0.0000	0.0000	0.0000	...	0.0629	0.0325	0.0152	0.0638	0.0645
14	0.0000	0.0000	0.0000	...	0.1235	0.0745	0.0396	0.0628	0.0383
15	0.0000	0.0000	0.0000	...	0.1948	0.1409	0.0879	0.0601	0.0192
16	0.0000	0.0000	0.0000	...	0.2353	0.2119	0.1594	0.0542	0.0078
17	0.0000	0.0000	0.0000	...	0.2023	0.2399	0.2278	0.0438	0.0026
18	0.0000	0.0000	0.0000	...	0.1092	0.1868	0.2403	0.0291	0.0006
19	0.0000	0.0000	0.0000	...	0.0292	0.0834	0.1663	0.0139	0.0001
20	0.0000	0.0000	0.0000	...	0.0029	0.0128	0.0567	0.0035	0.0000
Total	1.0000	1.0000	1.0000	...	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	3.5369	4.0253	4.5256	...	15.7106	16.4514	17.1922	9.9568	9.2848
E(Ym)	0.2110	0.3352	0.5137	...	10.7106	11.4514	12.1922	5.2352	4.3635
E(Zm)	0.2694	0.3668	0.4759	...	1.0000	1.0000	1.0000	0.8544	0.9488
E(Am)	0.1768	0.2013	0.2263	...	0.7855	0.8226	0.8596	0.4978	0.4642
E(Bm)	0.0872	0.1300	0.1847	...	0.2145	0.1774	0.1404	0.3291	0.4566

Table 1.63 : Probabilities $P_m(n)$, $Q_m(n)$, $P(n)$ and some important performance measures for Geom(n)/Geom(n)/C/N machine interference problems with C=5, N=20, m=10, ϵ= 0.4, μ=0.1 i = 0,1,2...................20, and $P_o(i)$ =1/21.

	0	1	2		18	19	20	$Q_m(n)$	P(n)
0	0.0003	0.0002	0.0001		0.0000	0.0000	0.0000	0.0000	0.0000
1	0.0065	0.0044	0.0029		0.0000	0.0000	0.0000	0.0008	0.0000
2	0.0478	0.0354	0.0255		0.0000	0.0000	0.0000	0.0072	0.0027
3	0.1651	0.1346	0.1073		0.0000	0.0000	0.0000	0.0305	0.0161
4	0.2953	0.2683	0.2376		0.0000	0.0000	0.0000	0.0708	0.0532
5	0.2789	0.2861	0.2837		0.0000	0.0000	0.0000	0.0956	0.1021
6	0.1448	0.1737	0.1988		0.0000	0.0000	0.0000	0.0903	0.1392
7	0.0493	0.0727	0.0993		0.0000	0.0000	0.0000	0.0807	0.1624
8	0.0106	0.0205	0.0350		0.0000	0.0000	0.0000	0.0742	0.1624
9	0.0013	0.0037	0.0084		0.0008	0.0001	0.0000	0.0715	0.1392
10	0.0001	0.0004	0.0013		0.0059	0.0012	0.0001	0.0707	0.1021
11	0.0000	0.0000	0.0001		0.0269	0.0089	0.0019	0.0705	0.0638
12	0.0000	0.0000	0.0000		0.0796	0.0374	0.0130	0.0703	0.0239
13	0.0000	0.0000	0.0000		0.1605	0.1022	0.0511	0.0694	0.0150
14	0.0000	0.0000	0.0000		0.2279	0.1896	0.1291	0.0656	0.0055
15	0.0000	0.0000	0.0000		0.2293	0.2442	0.2191	0.0563	0.0019
16	0.0000	0.0000	0.0000		0.1620	0.2183	0.2537	0.0408	0.0004
17	0.0000	0.0000	0.0000		0.0782	0.1328	0.1983	0.0229	0.0001
18	0.0000	0.0000	0.0000		0.0243	0.0522	0.1002	0.0092	0.0000
19	0.0000	0.0000	0.0000		0.0043	0.0119	0.0296	0.0023	0.0000
20	0.0000	0.0000	0.0000		0.0003	0.0012	0.0039	0.0003	0.0000
Total	1.0000	1.0000	1.0000	...	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	4.4846	4.7349	5.0022	...	14.4807	15.1456	15.8104	9.4599	7.7358
E(Ym)	0.2809	0.3972	0.5432	...	9.4807	10.1456	10.8104	4.6170	2.8301
E(Zm)	0.4851	0.5571	0.6266	...	1.0000	1.0000	1.0000	0.8906	0.9278
E(Am)	0.2242	0.2367	0.2501	...	0.7240	0.7573	0.7905	0.4730	0.3868
E(Bm)	0.1406	0.1833	0.2300	...	0.2760	0.2427	0.2095	0.3654	0.4770

Table 1.64 : probabilities $P_t(n)$, $Q_t(n)$, (Pn) and some important performance measures for a m/m/c/N machine interference with t=10 and rest of the parameters as in table 1.63

	0	1	2		18	19	20	$Q_m(n)$	P(n)
0	0.0078	0.0054	0.0037	...	0.0000	0.0000	0.0000	0.0011	0.0006
1	0.0429	0.0325	0.0243	...	0.0000	0.0000	0.0000	0.0073	0.0044
2	0.1115	0.0923	0.0750	...	0.0000	0.0000	0.0000	0.0228	0.0167
3	0.1820	0.1634	0.1434	...	0.0000	0.0000	0.0000	0.0456	0.0402
4	0.2080	0.2008	0.1892	...	0.0000	0.0000	0.0000	0.0657	0.0683
5	0.1736	0.1793	0.1805	...	0.0002	0.0001	0.0000	0.0720	0.0874
6	0.1247	0.1381	0.1493	...	0.0005	0.0002	0.0001	0.0746	0.1049
7	0.0774	0.0902	0.1071	...	0.0015	0.0006	0.0003	0.0747	0.1175
8	0.0415	0.0530	0.0664	...	0.0040	0.0019	0.0008	0.0737	0.1222
9	0.0193	0.0262	0.0356	...	0.0099	0.0050	0.0024	0.0725	0.1173
10	0.0077	0.0112	0.0163	...	0.0225	0.0124	0.0063	0.0715	0.1032
11	0.0026	0.0041	0.0064	...	0.0457	0.0275	0.0153	0.0706	0.0826
12	0.0008	0.0013	0.0021	...	0.0820	0.0546	0.0335	0.0693	0.0594
13	0.0002	0.0003	0.0006	...	0.1281	0.0954	0.0650	0.0669	0.0380
14	0.0000	0.0001	0.0001	...	0.1706	0.1441	0.1104	0.0624	0.0213
15	0.0000	0.0000	0.0000	...	0.1887	0.1837	0.1607	0.0544	0.0102
16	0.0000	0.0000	0.0000	...	0.1668	0.1912	0.1950	0.0430	0.0041
17	0.0000	0.0000	0.0000	...	0.1115	0.1548	0.1893	0.0291	0.0013
18	0.0000	0.0000	0.0000	...	0.0518	0.0902	0.1378	0.0157	0.0003
19	0.0000	0.0000	0.0000	...	0.0144	0.0329	0.0669	0.0059	0.0001
20	0.0000	0.0000	0.0000	...	0.0018	0.0054	0.0162	0.0012	0.0000
Total	1.0000	1.0000	1.0000	...	1.0000	1.0000	1.0000	1.0000	1.0000
E(Xm)	4.4244	4.7178	5.0294	...	14.5384	15.2087	15.8790	9.5082	8.0480
E(Ym)	0.5419	0.6793	0.8460	...	9.5384	10.2087	10.8790	4.7681	3.2672
E(Zm)	0.4477	0.5056	0.5644	...	0.9999	1.0000	1.0000	0.8575	0.8698
E(Am)	0.2212	0.2359	0.2515	...	0.7269	0.7604	0.7940	0.4754	0.4024
E(Bm)	0.1784	0.2108	0.2456	...	0.2729	0.2395	0.2060	0.3507	0.4235

CHAPTER - 2

DISCRETE TIME TRANSIENT SOLUTION FOR BUSY PERIOD ANALYSIS, DOUBLE ENDED QUEUEING AND MACHINE INTERFERENCE MODEL

CHAPTER - TWO

DISCRETE TIME TRANSIENT SOLUTION FOR BUSY PERIOD ANALYSIS, DOUBLE ENDED QUEUEING AND MACHINE INTER-FERENCE MODEL

2.1 DISCRETE TIME TRANSIENT SOLUTION FOR BUSY PERIOD ANALYSIS

INTRODUCTION

Busy period analysis from an integral part of any queueing system as the distribution of its duration is important from several point of view. It is also helpful in the efficient planning of the system. A busy period is initiated with the arrival of customer who finds the system empty and ends when the system next becomes force. However the initial busy period may be initiated with any number i customers in the system Busy period analysis concerning simple Markovian queueing system has been carried under two basic assumptions namely the system is taken in 'empty' and 'idle' initial condition and the system has infinite waiting room capacity. No attempt seems to have been made to obtained a transient solution for such a queueing system in discrete time.

This chapter provides transient solution in discrete time for busy period analysis in Queueing theory under arbitrary initial condition and finite waiting space. It is also shown that the results in continuous time can be easily obtained in explicit closed form expressions are obtained in terms of roots. Extensive numerical results are also given using the closed form expression.

ASSUMOTION –

1. The queue consists of finite waiting space .
2. Inter arrival and service propabities are independent of the system .
3. Inter-arrival and service time distributions are geometric but independent of time .
4. Queue discipline is first in first out (F.I.F.O)

NOTATION-

X_k : Number of customers in the system at epoch k

N : Size of the queue (maximum) .

λ_m : Inter-arrival probabilities when n customers are in the system.

μ_n : Service probabilities when n customers are in the system

ψ_n : $\lambda_n(1-\mu_n)$ ϕ_n: $\mu_n\,(1-\lambda_n)$.

ANALYSIS OF THE MODEL :-

We define i-channel busy period $0 < i \le N$ to begin with an arrival to the system at an epoch m, when there are (i-1) castomers in the system to the very next epoch (m+1), when there are again (i-1) castomers in the system.

The following differential-difference equations may be defined as

$$P_{m+1}(i-1) - P_m(i-1) = \psi_i\, P_m(i) \qquad \text{........(2.1)}$$

$$P_{m+1}(i) - P_m(i) = -(\psi_i+\phi_i)\, P_m(i) + \phi_{i+1}\, P_m(i+1) \qquad \text{........(2.2)}$$

$$P_{m+1}(n) - P_m(n) = -(\psi_n+\phi_n)\, P_m(n) + \psi_{n-1}\, P_m(n-1) + \phi_{n+1}\, P_m(n+1) \qquad i<n\le N-1 \qquad \text{........(2.3)}$$

$$P_{m+1}(N) - P_m(N) = -\phi_N\, P_m + \psi_{N-1}\, P_m(N-1) \qquad \text{........(2.4)}$$

Let $P_z(n)$ be the probability generating function (p.g.f.) of $P_m(n)$ defined as.

$$G(z,n) = \sum_{m=0}^{\infty} z^m\, P_m(n); \qquad |z| \le 1 \qquad \text{........(2.5)}$$

Taking the p.g.f of equation (2.1) to (2.4), for this we multiply the equation by z^m and taking summation from 0 to ∞ for m and using $(1-z)/z = s$ we get

$$sP_Z(i-1) - \phi i\, P_Z(i) = 1/z \qquad \text{........ (2.6)}$$

$$(s+\phi_i+\psi_i)\, P_Z(i) - \phi_{i+1}\, P_Z\,(i+1) = 1/z \qquad \text{........ (2.7)}$$

$$-\psi_{n-1}\, P_Z(n-1) + (s+\psi_n+\phi_n)\, P_Z(n) - \phi_{N+1}\, P_Z(n+1) = 1/z \qquad \text{........ (2.8)}$$

$$-\psi_{n-1}\, P_Z(N-1) + (s+\phi_N)\, P_Z(N) = 1/z \qquad \text{........ (2.9)}$$

We get

$$AP = [\delta_k(i-1),\, \delta_{ki},\, \delta_{k(i+1)}, \ldots\ldots\ldots\, \delta_{kN}]' \qquad \text{.....(2.10)}$$

Where A is real tri-diagonal matrix of order (N-i+2) x (N-i+2) and P is a column vector of δ_{ki} is a Kronecker delta defined as

$$\delta_{ki} = \left\{ \begin{array}{lll} 1/z & ; & k=i \\ 0 & ; & \text{otherwise} \end{array} \right\}$$

Defining $s = (1-z)/z$, we get

$$A(s)= \begin{array}{c} \\ i-1 \\ i \\ i+1 \\ . \\ . \\ . \\ N-1 \\ N \end{array} \begin{array}{c} \begin{array}{cccccccc} i-1 & i & i+1 & i+2 & - & N-2 & N-1 & N \end{array} \\ \left[\begin{array}{cccccccc} s & -\psi_i & 0 & 0 & - & 0 & 0 & 0 \\ 0 & s+\phi_i+\psi_i & -\phi_i+1 & 0 & - & 0 & 0 & 0 \\ 0 & -\psi_i & s+\phi_i+1 & -\phi_i+2 & - & 0 & 0 & 0 \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ - & - & - & - & - & - & - & - \\ 0 & 0 & 0 & 0 & - & -\psi_{N-2} & s+\phi_{i\,N-1}+\psi_{N-1} & -\phi_N \\ 0 & 0 & 0 & 0 & - & 0 & -\psi_{N-1} & s+\phi_N \end{array} \right] \end{array}$$

(N-i+2) x (N-i+2)

$$P_z(n) = \begin{bmatrix} P_z(i) \\ P_z(i+1) \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ P_z(N) \end{bmatrix}_{(N-i+2) \times 1}$$

From equation (2.10), using Cramer's rule $P_Z(n)$ are explicitly determined as

$$P_z(n) = \frac{|A_{n-i+2}(s)|}{|A(s)|} \qquad i-1 \leq N$$

Where A_{n-i+2} is obtained from A(s) by replacing the $(n-i+2)^{th}$ column of A(s) by R.H.S. of eqn. (2.10) and $|A(s)|$ is the determinant of A(s).

Applying some row and column transformations on $|A(s)|$, it may be expressed as $|D(s)|$ where D(s) is a real, symmetric, tridiagonal matrix of order (N-i+1) x (N-i+1).

$$D(s) = \begin{bmatrix} s+i & -\sqrt{\psi_i \phi_i} & 0 & \cdots\cdots & 0 & 0 & 0 \\ -\sqrt{\psi_i \phi_i} & s+\phi_i+\psi_I & -\sqrt{\psi_{i+1}\phi_{i+1}} & \cdots\cdots & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots\cdots & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdots\cdots & & \cdot & \cdot \\ 0 & 0 & 0 & \cdots\cdots & -\sqrt{\psi_{N-2}\phi_{N-2}} & s+\phi_{N-2}+\psi_{N-2} & -\sqrt{\psi_{N-1}\phi_{N-1}} \\ 0 & 0 & 0 & \cdots\cdots & 0 & -\sqrt{\psi_{N-1}\phi_{N-1}} & s+\phi_{N-2}+\psi_{N-2} \end{bmatrix}_{(N-i+1) \times (N-i+1)}$$

$|D(s)|$ are the negatives of the eigen values of the matrix D(0). It may be observed that D(0) is a positive definite symmetric tri-diagonal matrix and its eigen values are positive, real and

distinct. Thus the roots of the polynomial $|A(s)|$ are real negative and distinct (one root is zero).

Let α_k (k=o,1,2,.......N=i+1) be the roots of $|A(s)|$ with k=0. Then

$$|A(s)| = s \prod_{k=1}^{N-i+1} (s - \alpha_k)$$

And hence $P_z(n) = \dfrac{|A_{n-i+2}(s)|}{s \prod_{k=1}^{N-i+1} (s - \alpha_k)}$; $\quad i - 1 \leq N$

Resolving right hand side of $P_z(n)$ into Partial fractions and replacing by (1-z)/z, using initial condition and comparing coefficient of z^m, we get.

$P_m(i-1)$ = Coff. of z^m in $P_Z(i-1)$

$$= \text{Coff. of } z^m \text{ in } \frac{|A_1(s)|}{s \prod_{j=1}^{N-i+1} (s - \alpha_j)}$$

$P_m(i-1) = 1+ \phi_i\, a_k(i-1)\, (1+\alpha_k)^m$

$P_m(i)$ = Coff. of z^m in $P_z(i)$

$$= \text{Coff. of } z^m \text{ in } \frac{|A_1(s)|}{s \prod_{j=1}^{N-i+1} (s - \alpha_j)}$$

Where

$P_m(i)$ = Coff. of z^m in $P_z(i)$

$$= \text{Coff. of } z^m \text{ in } \frac{|A_{n-2+1}(s)|}{s \prod_{j=1}^{N-i+1} (s - \alpha_j)}$$

$$= s \prod_{r=1}^{n-1} \psi_{r+1} \sum_{k=1}^{N-i+1} a_{kn} (1+\alpha_k)^m; \qquad i \leq 1 \leq N-1$$

Where

$$a_k(i-1) = \frac{C_{N-1}(\alpha_k)}{\alpha_k \prod_{\substack{j=1 \\ j \neq k}}^{N-i+1} (\alpha_k - \alpha_j)}$$

$$a_{ki} = \frac{C_{N-1}(\alpha_k)\, D_1(\alpha_j)}{\alpha_k \prod_{\substack{j=1 \\ j \neq k}}^{N-i+1} (\alpha_K - \alpha_k)}$$

Where $C_n(s)$ and D_n being the determinants obtained by the bottom right and top left (nxn) square matrices farmed from A(s) such that.

$$|A(s)| = C_{N-i+2}(s) = D_{N-i+2}(s).$$

It may be remarked that for the probabilities $P_m(n)$, $i-1 \leq n \leq N$ to remain bounded $|1+\alpha_k| < 1$ which is true if $\phi_k + \psi_{k-1} < 1$. Under this condition, the sum of the absolute value of the elements in each row of the matrix D(0) is less than 2 and hence from Gerschgorin's Theorem $|\alpha_k| < 2$; Hunter [6], which implies $|1+\alpha_k| < 1$.

$C_n(s)$ and $D_n(s)$ may be determined by the following recurrence relations.

Assuming $C_0(s)$ and $D_0(s) = 1$

$$C_1(s) = s+\phi_N,\ D(s) = s$$

And $\lambda = \psi_{i-1} = \phi_{i-1} = 0$

$$C_k(s) = (s+\psi_{N+1+k} + \phi_{N-1-k})\, C_{k-1} - \psi_{N+1-k}\, \phi_{N+2-k}\, c_{k-2} \qquad ;2 \leq k \leq N-i+2$$

$$D_k(s) = (s+\psi_{i+k-2} + \phi_{i+k-2})\, D_{k-1} - \psi_{i+k-3}\, \phi_{i+k-2}\, c_{k-2} \qquad ;2 \leq k \leq N-i+2$$

Using the standard IMSL packages one can find the eigen values and hence the verse of polynomial $|A(s)|$. Further using recurrence relation for $C_k(s)$ and $D_k(s)$ can easily be evaluated.

Using IMSL to find eigen values for large N would require much larger memory, hence for greater precision, for large N(N>200) one might be force to use the main frame computer. The first author has also the experience of working with QROOT developed at RMC, Canoda see[4].

Therefore some comment on QROOT will be in order. The accuracy of the roots by QROOT given by $|A(s)|<10^{-14}$ is not sufficient for problems under consideration because of the recurrence relation involved in finding the roots and the probabilities $P_m(n)$. IMSL proves better than QROOT for the problem considered in this section.

Since $(1+\alpha_k)^m \to 0$ as $m \to 0$ the steady state distribution is given by

$P(i-1) = 1;$

$P(n) = 0 \qquad\qquad i \leq n \leq N$

The solution presented here is expressed as the sum of two parts one pertainting to the steady state and the other to the transient.

NUMERICAL RESULTS :

We given below the numerical results for both the discrete and continuous cases of the model discussed above for the sake of convenience, the results for only moderate values of N are given, though there were no problems even for large values of N.

Assume $r=6$, $N=20$, $\lambda=0.8$, $\mu=0.15$, $m=10$, $i=5$ & 6

The following table gives the probabilities $P_m(n)$ and the steady state probabilities.

i =5

n	$P_m(n)$	P(n)
4	0.5507	1.0000
5	0.1367	0.0000
6	0.2083	0.0000
7	0.0798	0.0000
8	0.0204	0.0000
9	0.0036	0.0000
10	0.0005	0.0000
11	0.0000	0.0000
12	0.0000	0.0000
13	0.0000	0.0000
14	0.0000	0.0000
15	0.0000	0.0000
16	0.0000	0.0000
17	0.0000	0.0000
18	0.0000	0.0000
19	0.0000	0.0000
20	0.0000	0.0000

i = 6

n	$P_m(n)$	P(n)
5	0.8524	1.0000
6	0.0108	0.0000
7	0.0186	0.0000
8	0.0227	0.0000
9	0.0231	0.0000
10	0.0208	0.0000
11	0.0169	0.0000
12	0.0127	0.0000
13	0.0088	0.0000
14	0.0057	0.0000
15	0.0035	0.0000
16	0.0020	0.0000
17	0.0010	0.0000
18	0.0005	0.0000
19	0.0003	0.0000
20	0.0002	0.0000

Finally the Discrete time models are very important for application purposes such as computer, it seems that this area of research has largely been ignored particularly when the transient solution are need.

2.2 DISCRETE TIME CLOSED FORM TRANSIENT SOLUTIONS FOR DOUBLE ENDED QUEUEING SESYTEM

The concept of double ended queue was introduced by Kendall (1951). The example of such a queueing system is a taxi stand where at times passengers queue up for taxis and at other time taxis wait for passengers .

Shrivastava and Kashyap (1982) obtained the transient solution of a double ended queue in terms of summation of the integrals of Bessel function, which is quite cumbersome and difficult to compute. However recently Sharma (1990) provided a simple algebraic closed farm expression and easy to compute for the transient probabilities in continuous time seems no attempt has been made to obtain similar results in discrete time.

It this section we obtain closed form transient solution far the double ended queueing system in discrete time .It is also further shown how the corresponding results in continuous time can be obtained. The work presented in this section is continuation with the earliear work done by kapur et al (1996).

ASSUMPTION –

(1) The numbers on the negative axis stand for taxis waiting far customers on passengers where as the numbers on the positive axis denote the passengers waiting for taxis

(2) The queue consist of finite waiting space.

(3) Probabilities of passengers arrival is λ and probabilities of taxis arrival is μ.

(4) Arrival probabilities of passengers and taxis fallows geometric distributions .

(5) The queue discipline is FCFS.

NOTATION-

X_k: Number of customers taxi at epoch k.

M : Maximum number of taxis

N : maximum number of customers .

λ : Arrival probabilities of passengers or customers.

μ : Arrival probabilities of taxi .

$\phi : \lambda(1-\mu)$, $\psi : \mu(1-\lambda)$.

ANALYSIS OF THE MODEL :

Let X_m be either the number of customers or taxis at discrete time epoch m then $\{X_k\}$, $k \geq 0$ is an integer valued discrete stochastic process.

Taking values $-M, -M+1, \ldots 0, 1, 2, \ldots N-1, N$, $X_k = k$ $(-M \leq k \leq N)$ implies that there are either M taxis or N customer waiting at epoch k . On each arrival or service the process X_k behaves as a discrete time Markov process and represent the state of the system.

The difference –differential equation may be written as

$$P_{m+1}(-M) = P_m(-M)(1-\lambda) + P_m[-(M-1)]\mu(\lambda) \qquad \ldots\ldots\ldots(2.11)$$

$$P_{m+1}[-(M-1)] = P_m[(M-1)][(1-\lambda)(1-\mu) + \lambda\mu] + P_m[-(M-2)]\mu(1-\lambda) + P_m(-M)\lambda \qquad \ldots\ldots\ldots(2.12)$$

$$P_{m+1}(k) = P_m(k)][(1-\lambda)(1-\mu) + \lambda\mu] + P_m[(k+1)]\mu(1-\lambda) + P_m(k-1)\lambda(1-\mu) \quad ; -(M-2) \leq k \leq N-2 \qquad \ldots\ldots\ldots(2.13)$$

$$P_{m+1}(N-1) = P_m(N-1)[(1-\lambda)(1-\mu) + \lambda\mu] + P_m(N)\mu + P_m(N-2)\lambda(1-\mu) \qquad \ldots\ldots\ldots(2.14)$$

$$P_{m+1}(N) = P_m(N)(1-\mu) + P_m(N-1)\lambda(N-1) \qquad \ldots\ldots(2.15)$$

With $P_0(-M) = P_0(i) = 0, -(M-1) < i < N$

Let $P_z(n)$ be the probabilities generating function (p.g.f) of $P_m(n)$ defined as

$$P_z(n) = \sum_{m=0}^{\infty} z^m P_m(n), \qquad |z| \le 1$$

Now taking the P.g.f of the equation (2.11) to (2.15) we have .

$$s P_z(-M) - \mu\ (1-\lambda)\ P_z[-(-M-1)] = 1/z \qquad \text{........(2.16)}$$

$$\lambda P_z(-M) + (s+\phi+\psi) P_z[(-M-1)] - \phi P_z[-(M-2)] = 0 \qquad \text{......(2.17)}$$

$$\phi P_z(k-1) + (s+\phi+\psi) P_z(k) - \psi P_z(k+1) = 0 \qquad \text{.....(2.18)}$$

$(M-2) \sum k \in N\text{-}z$

$$\phi P_z(N-2) + (s+\phi+\psi) P_z(N-1) - \mu P_z(N) = 0 \qquad \text{............(2.19)}$$

$$\phi P_z(N-1) + (S+\mu) P_z(N) = 0 \qquad \text{............(2.20)}$$

$$AP = [\delta_{0(-M)}, \delta_{0(-M+1)} \ldots\ldots\ldots \delta_{00}, \delta_{01}, \ldots\ldots\ldots\ldots\ldots \delta_{0N}] \qquad \text{............(2.21)}$$

Where A is a real tridiagonal matrix of order $(N\text{-}k-2) \times (N\text{-}k-2)$; P is a column vector and δ_{0k} the Kronecker delta defined as

$$\delta_{0k} = \left\{ \begin{array}{ll} 1/z\ ; & k = -M \\ \\ 0\ ; & \text{otherwise} \end{array} \right\}$$

$$P = \begin{bmatrix} P_z(-M) \\ P_z(-M+1) \\ P_z(-M+2) \\ \vdots \\ \vdots \\ P_z(-1) \\ P_z(0) \\ P_z(1) \\ \vdots \\ P_z(N-2) \\ P_z(N-1) \\ P_z(N) \end{bmatrix}_{(N+M+1) \times 1}$$

Using Cramer's rule

$$P_z(N) = \frac{|A_{N+M+1}(s)|}{|A(s)|} \quad ; \quad |z| \leq 1 \qquad \text{..........(2.22)}$$

where $A_{N+M+1}(s)$ is obtained from matrix A by replacing the $(N+M+1)^{th}$ column by right hand side of (2.21) and $|A(s)|$ is the determinant of $A(s)$

Observe that D(0) is a positive definite symmetric tri-diagonal matrix , therefore its eigen values are real, positive and distinct Kijima (1992). Hence the roots of the polynomial $|A(s)|$ are real negative and distinct (one root being zero).

Let α_i $(i = 0,1,2,\ldots\ldots\ldots, N+M+1)$ be the roots of $|A(s)|$ with $(\alpha_0 = \alpha_{-M} = 0)$ then

$$|A(s)| = s \prod_{i=1}^{N+M+1} (s - \alpha_i)$$

And hence

$$P_z(N) = \frac{|A_{N+M+1}(s)|}{s \prod_{i=1}^{N+M+1} (s - \alpha_i)}$$

$\alpha_{-M} = \alpha_0$, $\alpha_{-M+1} = \alpha_1$, $\alpha_{-M+2} = \alpha_2$, ...$\alpha_{N-2} = \alpha_{N+M-2}$,

$\alpha_{N-1} = \alpha_{N+M-1}$, $\alpha_N = \alpha_{N+M+1}$,

Resolving equation (2.22) into partial fraction and replacing s by $(1-z)/z$ and using initial condition and comparing coefficient of z^m we have ,

$$P_m(N) = (-1)^{N+M} \lambda(\phi)^{N+M-1} \left[\prod_{i=1}^{N+M+1} (-\alpha_i)^{-1} + \prod_{i=1}^{N+M+1} (\alpha_i)^{-1} + \frac{\left(\prod_{j=1, \neq I}^{N+M+1} (\alpha_i - \alpha_j)\right)^{-1}}{(1+\alpha_j)^m} \right]$$

..........(2.23)

however, if one is interested in finding the value of $P_m(N)$ under are be arbitrary initial condition, one may obtain the probability $P_m(N)$ as

$$P_m(N) = \prod_{j=0}^{M+N} \frac{\phi}{(-\alpha_j+1)} + (\phi)^{N+M} \sum_{j=1}^{N+M+1} a_j (1+\alpha_j)^{m-} \qquad \ldots\ldots(2.24)$$

where

$$a_j = \frac{D_i(\alpha_j)}{\alpha_j \prod_{i=1, \neq j}^{N+M+1} (\alpha_i - \alpha_j)}$$

and

$$P_m(N) = \prod_{j=0}^{M+N} \frac{\phi}{(-\alpha_j+1)} + \sum_{j=1}^{N+M+1} a_j (1+\alpha_j)^{m-} \qquad \text{if } i = N+M+1 \qquad \ldots\ldots\ldots(2.25)$$

where D_i (s) being the determinant obtained by the top left (i×i) square matrix formed from A(s) such that $|A(s)| = D_{N+M+2}(s)$, $\lambda_N = \lambda_{N+M+1} = 0$, D_i (s) is obtained by the following recursive relation which can be easily obtained because of tri-diagonal matrix of $|A(s)|$.

Assume $D_o(s) = 1$, $D_1(s) = S + \lambda$, $D_{-M} = 0$ and $\lambda N = 0$

$D_i(s) = (s + \phi + \psi)\ D_{i-1}(s) - \phi_{i-2}\, \psi_{i-1}\, D_{i-1}(s)\ ,\ z \leq i \leq N + M$

To get the roots α_i (i = 1,2, 3 ………., N, M+1), we have used the IMSL package. It may be noted that we find the eigen values of the matrix D (0) and roots are negatives of the eigen values. The accuracy of the roots equal has been verified against the request in algebra viz, sum of the negatives of the roots is equal to the to the sum of the elements on the principal diagonal of A(0). The routines of the package are quite efficient and produce result to a high degree of accuracy even when the matrix size is greater so. Since $(1+\alpha_i)^m \rightarrow 0$ as $m \rightarrow \infty$, the steady state distribution is given by

$$P_m(N) = \prod_{i=0}^{M+N} \frac{\phi}{(-\alpha_j + 1)}$$

Continuous Case :-

Letting $\lambda = \lambda(\Delta) + 0(\Delta)$;

$\mu = \mu(\Delta) + 0(\Delta)$

Taking m = t and m+1 = t+D, in the difference differential equations one can transform equation in t we can then proceed to get the continuous time solutions of the transformed equations. Alternatively, one can change the root equation and them get the continuous time solution form the final discrete time solution proceeding this way the roots, α_i of $|A(s)|$ are transform to $\alpha_i \Delta$. It is then easy to see that $(1+\alpha_i\Delta)^m$ tends to $e_i^{\alpha+1}$ in continuous time, where t is divided into m such interval each of length Δ such that $t = m\Delta$.

Now treating λ and μ as the arrival rate of passenger and arrival rate of taxi respectively. One can get the transient solution of the continues model. Note that the roots of $|A(s)|$ were formed in $s = (1-z)/z$. Parameters in continues case may then be treated as the transform parameter. Now the R.H.S of matrix equation will have 1 in the i^{th} place instead of 1/z. Analytic and numerical results in this also be verified as given in literature.

The analogy gives the fact that discrete time model discussed in this section are more general and correspondingly provide results in continues time with a real case.

$$P_m(N) = (-1)^{N+M} \lambda(\phi)^{N+M-1} \left[\left\{ \prod_{i=1}^{N+M+1} (-\alpha) \right\}^{-1} + \sum_{i=1}^{N+M+1} \left\{ (\alpha_i)^{-1} \prod_{j=1, j \neq i}^{N+M+1} (\alpha_i - \alpha_j)^{-1} \right\} e^{\alpha it} \right]$$

2.3 DISCRETE TIME CLOSED FROM TRANSIENT SOLUTION FOR GEOM (n)/ GEOM (n) /1/N MACHINE INTERFERENCE MODEL

Benson and Cox (1951) were the first to formulate an M/M/r machine interference problem incorporating the operative efficiency and machine availability and they have obtained steady-state solution : Later on Benson (1951) Maritas and Xirokostas (1977) and Bunday and Scraton (1980) have also investigated state –state measure of effectiveness. However, in most practical situations where the machine interference model is applicable such as textile industries, robot operated factories and large computer center the study of transient solution are rather unavoidable if the true working of these system to be monitored over a small horizon of time Recently Sharma (1990) obtained simple closed farm expressions for state probabilities for such a system in continuous time. In this section we obtain closed form transient solution for Geom(n)/Geom(n)/1/N machine interference model in discrete time. It is also shown how the corresponding results in the continuous time can be obtained. Further we discussed multi –operative system wit an arbitrary initial number of machine interference. Extensive numerical results are also give using the also form expressions for multi server interference model

ASSUMPTION :

(1) The queue consists of finite waiting space .

(2) Probability of a machine breakdown is $\lambda$and probability of repair a machine is μ.

(3) Probability of machine are break downs and repairs follows geometric distribution

(4) The queue discipline is FCFS.

NOTATION :

X_k: Number of machine breakdowns at epoch k.

N: Maximum number of machines .

λ: probability of machine breakdown .

μ: probability of machine repair.

λ_n: (N-n)λfar all n=0;1;2;.......N.s

μ_n : μ for all n = 1, 2,.........N with $\mu_0 = 0$, $\lambda_N = 0$

ψ_n : $\lambda_n (1 - \mu_n)$ with $\psi_0 = \lambda_0 = N\lambda$, $\psi_N = 0$

ϕ_n : $\mu_n (1 - \lambda_n)$ with $\phi_0 = 0$, $\phi_1 = \mu [1-(N-1) \lambda]$

ANALYSIS OF THE MODEL :

Let X_n be the number of machine breakdown at discrete time epoch m then { $X_k : k \geq 0$ } is an integer valued discrete stochastic process

Taking values 0,1,2......N . $X_k = k$ implies that there are k machine breakdowns at epoch k . Here arrival refers to the repair of a machine. The process { X_k }behaves as a discrete time Markov – Process and represents the state of the system .

Let $P_m(n)$ denote the probability that the system is in the n th state at the beginning of the m th epoch i.e

$P_m(n)$ = probability that there are n machine breakdowns at a given epoch m , $m \geq 0$. We further assume that breakdowns at a given epoch m, $m \geq 0$. We further assume that i breakdowns initially need attention at m=0.

Then $P_0(n) = \delta_{in}$

$$P_0(i) = 1 \qquad ; 0 \leq i \leq N$$

The difference – differential equations may be written as

$$P_{m+1}(i) = P_m(0)\,(1\text{-}N\lambda) + P_m(1)\mu\,[1\text{-}(N\text{-}1)\,\lambda] \qquad \text{.......(2.26)}$$

$$P_{m+1}(i) = P_m(1)\,(1\text{-}\mu)\,[1\text{-}(N\text{-}1)\,\lambda] + P_m(0)\,N\lambda + P_m(2)\,\mu\,[1\text{-}(N\text{-}2)\,\lambda] \qquad \text{....... (2.27)}$$

$$P_{m+1}(i) = P_m(n)\,[1\text{-}(N\text{-}1)\,\lambda]\,(1\text{-}\mu) + P_m(n\text{-}1)\,(N\text{-}n\text{+}1)\lambda\,(1\text{-}\mu) \qquad \text{....... (2.28)}$$

$$P_{m+1}(N) = P_m(N)\,(1\text{-}\mu) + P_m(N\text{-}1)\,\lambda\,(1\text{-}\mu) \qquad \text{....... (2.29)}$$

Let $P_z(n)$ be the probability generating function (p.g.f) of $P_m(n)$ defined as

$$P_z(n) = \sum_{m=0}^{\infty} z^m\,P_m(n); \qquad |z| \le 1 \qquad \text{.........(2.30)}$$

Taking p.g.f. of equation (2.26) to (2.29) for this we multiply by z^m and taking summation from 0 to ∞ and $(1\text{-}z)/z = s$, we have

$$(s\text{+}N\lambda)\,P_z(0) - \mu[1\text{-}(N\text{-}1)\lambda]\,P_z(1) = 0 \qquad \text{....... (2.31)}$$

$$\text{-}N\lambda\,Pz(0) + [s\text{+}\mu + (N\text{-}1)\lambda\,(1\text{-}\mu)\text{+}\,(N\text{-}1)\,\lambda]\,P_z(1) - \mu\,[1\text{-}(N\text{-}2)\,\lambda]\,P_z(2) = 0 \qquad \text{.......(2.32)}$$

$$\text{-}[(N\text{-}n\text{+}1)\lambda]\text{+}(1\text{-}\mu)P_z\,(n\text{-}1) + [s\text{+}\mu\text{+}(N\text{-}n)\,(1\text{-}\mu)]\,P_z(n)\text{-}\mu[1\text{-}(N\text{-}n\text{-}1)\,\lambda]\,P_z\,(n\text{+}1) = 0$$

$$; n>1,\ n \ne i \qquad \text{........(2.33)}$$

$$\text{-}[(N\text{-}i\text{+}1)\lambda]\text{+}(1\text{-}\mu)P_z\,(i\text{-}1) + [s\text{+}(N\text{-}i)\,\lambda\,(1\text{-}\mu)]\,P_z(i)\text{-}\mu[1\text{-}(N\text{-}i\text{+}1)\,\lambda]\,P_z\,(i\text{+}1) = 1$$

$$\text{When } n = 0 \qquad \text{....... (2.34)}$$

$$\text{-}\lambda(1\text{-}\mu)P_z\,(N\text{-}1) + [s\text{+}\mu)]\,P_z(N) = 0 \qquad \text{....... (2.35)}$$

The above equations can be written in matrix notation as follows,

$$AP = [\delta_{io}, \delta_{i1}, \ldots\ldots\ldots\ldots\ldots\ldots \delta_{iN}]' \qquad \text{...........(2.36)}$$

Where P is a column vector given as

$$P = \begin{bmatrix} P_z(0) \\ P_z(1) \\ . \\ . \\ . \\ P_z(N-1) \\ P_z(N) \end{bmatrix}_{(N+1) \times 1}$$

and δ_{in} is a Kroneckar delta defined as

$$\delta_{in} = \begin{cases} 1/z & ; \quad \text{if } n=1 \\ \infty & ; \quad \text{otherwise} \end{cases} \qquad \text{........(2.37)}$$

Using the following notations,

Using $\psi_n = \lambda_n (1-\mu_n)$, $\psi_0 = \lambda_0 = N\lambda$, $\psi_N = 0$

$\phi_n = \mu_n(1-\lambda_n)$, $\phi_0 = 0$, $\phi_1 = [1-(N-1)]\lambda$

from equation (2.36) using Cramer's Rule we may determine $P_z(n)$ explicitely as

$$P_z(n) = \frac{|A_{n+1}(s)|}{|A(s)|}, \qquad 0 \leq n < N \qquad \text{.......(2.38)}$$

Where $A_{n+1}(s)$ is obtained from A(s) replacing the $(n+1)^{th}$ column of A(s) by the right hand side of equation (2.36) and $|A(s)|$ is the determinant of A(s).

Applying some row and column transformation $|A(s)|$. It may be expressed as s D(s), where D(s) is a real, symmetric, tri-diagonal matrix of N x N.

$|D(s)|$ is a polynomial of degree in N is S. It may be noted that D(0) is a positive definite symmetric, tri-diagonal matrix. It is well known that its eigen values are positive, real and distinct, one root being zero.

We observe that $D_n(s)$ satisfies the recurrence relation

$$D_n(s) - (s+\psi_{N-1} + \phi_{N-n+1}) D_{n-1}(s) + \psi_{N-n+1}\, \phi_{N-n+1}\, D_{n-2}(s) = 0 ; \qquad 1 \leq n \leq N$$

Let $\alpha_1, \alpha_2, \ldots\ldots \alpha_N$ be the roots of $D_n(s)$. Thus

$$P_z(n) = \frac{|A_{n+1}(s)|}{\prod_{i=1}^{N} (s - \alpha_i)}; \qquad 0 \le n < N \qquad \ldots\ldots(2.39)$$

Resolving the right hand side of $P_z(n)$ into partial fractions, by $(1-z)/z$ and comparing the coeff. of z^m, we have

$$P_m(n) = b_n + 1/n!\, \mu^{-n} \prod_{k=1}^{i} \mu\, B_n \sum_{k=1}^{N} a_{mn} (1-\alpha_k)^m; \quad n = 0 \qquad \ldots\ldots (2.40)$$

$$P_m(n) = b_n + \prod_{k=1}^{i} \mu\, B_n \sum_{k=1}^{n} a_{kn} (1-\alpha_k)^m; \qquad 1 \le n < i \qquad \ldots\ldots (2.41)$$

Where $B_n = \prod_{j=n}^{i-1} \{1-(N-j-1)\lambda$

$$P_m(n) = b_n + \prod_{k=1}^{n} a_{kn} (1-\alpha_k)^m; \qquad n = i \qquad \ldots\ldots (2.42)$$

$$P_m(n) = b_n + \{(n-i)!/(N-n)!\}\lambda^{n-1} \prod_{k=1}^{n-1} (1-\mu) \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m; \quad i+1 \le n < N \ldots\ldots(2.43)$$

Where a_{kn}'s are defined as

$$a_{kn} = \frac{C_{N-i}(\alpha_k)\, D_N(\alpha_k)}{\alpha_k \prod_{j=1\neq k}^{N} (\alpha_k - \alpha_j)}, \qquad 0 \le n \le i$$

$$a_{kn} = \frac{C_{N-n}(\alpha_k)\, D_1(\alpha_k)}{\alpha_k \prod_{j=1\neq k}^{N} (\alpha_k - \alpha_j)}, \qquad 0 \le n \le N$$

With $C_n(s)$ and $D_n(s)$ being the determinants obtained by the bottom right and top left (nxn) square matrices formed from A(s) such that

$$|A(s)| = C_{N+1}(s) = D_{N+1}(s)$$

$C_n(s)$ and $D_n(s)$ may be determined by the following relation.

$$C_n(s) = (s+\phi_{N-n+1} + \psi_{N-n+1})\, C_{n-1}(s) - \psi_{N-n+1}\, \phi_{N-n+2}\, C_{n-2}(s) \qquad ; 1 \le n \le N+1 \quad ......(2.44)$$

With $C_0(s) = 1$, $C_{-1}(s) = 0$

$$D_n(s) = (s+\phi_{n-1} + \psi_{n-1})\, D_{n-1}(s) - \phi_{n-1}\, \psi_{n-2}\, D_{n-2}(s) \qquad ; 1 \le n \le N+1 \quad ....(2.45)$$

With $D_0(s) = 1$ and $D_{-1}(s) = 0$

$$b_0^{-1} = \begin{cases} 1 + \sum_{r=1}^{N} {}^{N}C_r (\lambda/\mu)^r \dfrac{\prod_{i=1}^{r} \{1-(-1)\,\mu\}}{\prod_{i=1}^{r} \{1-(N-i)\,\lambda\}} & ; \quad n=1 \\ 1 + N(\lambda/\mu) \dfrac{1}{\{1-(N-1)\,\lambda\} \sum_{r=2}^{N} \dfrac{N!\,\lambda r\,(1-\mu)^{r-1}}{N^r (N-r)!\,(\mu)^r \prod_{i=1}^{r} \{1-(N-i)\,\lambda\}}} & \end{cases} \quad ....(2.46)$$

$$b_r b_0^{-1} = \begin{cases} {}^{N}C_r (\lambda/\mu)^r \dfrac{\prod_{i=1}^{r} \{1-(-1)\,\mu\}}{\prod_{i=1}^{r} \{1-(N-i)\,\lambda\}} & ; \quad i \le r \le 1 \\ N!/(N-r)!(\lambda/N\mu)^r \dfrac{(1-\mu)^r}{\prod_{i=1}^{r} \{1-(N-i)\,\lambda\}} & ; \quad 2 \le r \le N \end{cases} \quad ....(2.47)$$

It may be noted that for probabilities $P_m(n)$ to remain bounded if $|1+\alpha_k|<1$ which is true if $(\phi_k + \psi_{k-1})<1$, under this condition the sum of the absolute value of the elements in each row of the matrix D(0) < 2.

Hence $|\alpha_k| < 2 = |1+\alpha_k| < 1$.

Using IMSL package, we can get the eigen value of D(0). The routiences of this packages are quite efficient and produce result to a high degree of accuracy even when the matrix size is large that is > 50.

Since $(1+\alpha_k)^m \rightarrow 0$ as $m \rightarrow \infty$ the steady distribution Pm(n) may be defined as

$$P(n) \quad = \quad \lim_{m\rightarrow\infty} P_m(n) = b_n; \qquad 0 \leq n \leq N$$

Discrete Transient Solution For Geom(N)/Geom(N)/C/N Multiserver Machine Interference Model :

This model is an extension of the discrete transient solution of machine interference problem discussed above, here instead of single server there are $c(1 \leq c \leq N)$. Under indentical assumption made for a single server, we have for this model.

$$\lambda_n = \begin{cases} (N-n)\,\lambda & \text{if } n \leq N \\ 0 & \text{if } n \geq N \end{cases}$$

$$\mu_n = \begin{cases} n\mu & \text{if } 0 \leq n \leq c \\ c\mu & \text{if } n \geq c \end{cases}$$

We assume that inter-arrival and service times have geometric with parameters λ and μ respectively.

Denoting the probability that the system is in state n at the m^{th} epoch as $P_m(n)$ $(0 \leq n \leq N)$.

The following difference equations may be easily written as.

$$P_{m+1}(0) = P_m(0)\,(1-N\lambda) + P_m(1)\mu\,[1-(N-1)\,\lambda] \qquad \ldots\ldots\ (2.48)$$

$$P_{m+1}(1) = P_m(1)\,(1-\mu)\,[1-(N-1)\,\lambda] + P_m(0)\,N\lambda + P_m(2)\,\mu\,[1-(N-2)\,\lambda] \ \ldots\ldots\ (2.49)$$

$$P_{m+1}(n) = P_m(n)\,[1-(N-n)\,\lambda]\,(1-n\mu) + P_m(n-1)\{(N-n+1)\lambda\}\ \{1-(n-1)\mu\} + P_m(n+1)$$
$$[1-(N-n-1)\,\lambda]\,(n+1)\,\mu\,; \qquad 1 < n < c \ \ldots\ldots\ldots(2.50)$$

$$P_{m+1}(c) = P_m(c)\,[1-(N-c)\,\lambda]\,(1-c\mu) + P_m(c-1)\,[1-(c-1)\,\mu]\,(N-c+1)\,\lambda + P_m(c+1)\,(c\mu)$$
$$[1-(N-c-1)\,\lambda] \qquad \ldots\ldots\ldots\ (2.51)$$

$$P_{m+1}(n) = P_m(n)\,[1-(N-n)\,\lambda]\,(1-c\mu) + P_m(n-1)\,(1-n+1)\,\lambda\,(1-c\mu) + P_m(n+1)\,(c\mu)$$
$$[1-(N-n-1)\,\lambda]\,; \qquad c \leq n \leq N-1 \ \ldots\ldots\ (2.52)$$

$$P_{m+1}(N) = P_m(N)\,(1-c\mu) + P_m(N-1)\lambda\,(1-c\mu); \qquad \ldots\ldots\ (2.53)$$

Taking p.g.f. of equation (2.48) to (2.53), for this we multiply by z^m and summing from 0 to ∞, after simplification, we have

$$(s+N\lambda)\,P_z(0) - \mu[1-(N-1)\,\lambda]\,P_z(1) = 0 \qquad \ldots\ldots\ (2.54)$$

$$-N\lambda\,P_z(0) + [s+\mu+(N-1)\,\lambda\,(1-\mu)]\,P_z(1) - \mu[1-(N-2)\,\lambda]\,P_z(2) = 0 \qquad \ldots\ldots\ (2.55)$$

$$-(N-n+1)\lambda\ [1-(n-1)]\ P_z(n-1) - [s+n\mu+\ (N-1)\ \lambda\ (1-n\mu)]\ P_z(n) - (n+1)\ \mu[1-(N-n+1)\ \lambda]$$
$$P_z(n+1) = 0; \qquad 1 < n < c \ \ldots\ldots\ (2.56)$$

$$-(N-c+1)\,\lambda\,[1-(c-1)\,\mu]\,Pz(c-1) + [s+c\mu+(N-c)\,\lambda\,(1-c\mu)]$$
$$P_z(c) - P_z(c+1)c\mu[1-(N-c-1)\,\lambda] = 0 \qquad \ldots\ldots\ (2.57)$$

$$-(N-n+1)\,\lambda\,(1-c\mu)\,P_z(n-1) + [s+c\mu+(N-n)\,\lambda\,(1-c\mu)]$$
$$P_z(n) - c\mu[1-(N-n-1)\,\lambda]\,P_z(n)\,(n+1) = 0 \qquad \ldots\ldots\ (2.58)$$

$$-\lambda\,(1\text{-}c\mu)\,P_z(N\text{-}1) + (s+c\mu)\,P_z(N) = 0 \qquad \ldots\ldots\ (2.59)$$

Using the notations

$$\psi_n = \lambda_n\,(1\text{-}\mu_n) = \begin{cases} (N\text{-}n)\lambda\,(1\text{-}n\mu) & \text{if } 0 \le n < c \\ (N\text{-}n)\lambda\,(1\text{-}c\mu) & \text{if } n \ge c \end{cases} \qquad \ldots\ldots\ (2.60)$$

With $\psi_0 = \lambda_n = N\lambda$ and $\psi_N = 0$ Since $\lambda_N = 0$

$$\phi_n = \mu_n\,(1\text{-}\lambda_n) = \begin{cases} n\mu\,[1\text{-}(N\text{-}n)\,\lambda] & \text{if } 0 \le n < c \\ c\mu\,[1\text{-}(N\text{-}n)\,\lambda] & \text{if } c \le n < N \end{cases} \qquad \ldots\ldots\ (2.61)$$

$\phi_N = \mu_N\,(1\text{-}\lambda_N) = c\mu$ Since $\lambda_N = 0$

$$(s+\psi_0)\,P_0(0) = \psi_1\,P_z(1) = 0 \qquad \ldots\ldots(2.62)$$

$$-\psi_0\,P_z(0) + (s+\mu+\psi_1)\,P_z(1) = 0 \qquad \ldots\ldots(2.63)$$

$$-\psi_{n-1}\,P_z(n\text{-}1) + (s+\psi_n+\mu_n)\ -\ \phi_2\,P_z(2)\,P_z(n) - \phi_{n+1}\,Pz(n) = 0 \qquad \ldots\ldots(2.64)$$

$$-\psi_{c-1}\,P_z(c\text{-}1) + (s+c\mu+\psi_c)\,P_z(c) - \phi_{c+1}\,P_z(c+1) = 0 \qquad \ldots\ldots(2.65)$$

$$-\psi_{n-1}\,P_z(n\text{-}1) + (s+c\mu+\psi_n\text{-}\phi_{c+1})\,P_z(n+1) = 0 \qquad \ldots\ldots(2.66)$$

$$-\psi_{N-1}\,P_z(N\text{-}1) + (s+c\mu)\,P_z(N) = 0 \qquad \ldots\ldots(2.67)$$

With $P_0(i) = 1$; $\qquad 0 \le i \le N$

The above equations can be written in matrix notation as

$$AP = [\delta_{i0}, \delta_{i1}, \ldots\ldots\ldots\ldots\ldots\delta_{iN}]' \qquad \ldots\ldots(2.68)$$

Where P is a column matrix of order (N+1) x 1

$$P = \begin{bmatrix} P_z(0) \\ P_z(1) \\ . \\ . \\ P_z(N\text{-}1) \\ P_z(N) \end{bmatrix}_{(N+1)\times 1} \qquad \ldots\ldots(2.69)$$

From equation (2.68), using Cramer's Rule, we determine $P_z(n)$ explicitely as

$$P_z(n) = \frac{|A_{n+1}(s)|}{|A(s)|}, \qquad 0 \le n < N \quad ....(2.70)$$

Where $A_{n+1}(s)$ is obtained form A(s) by replacing the $(n+1)^{th}$ column A(s) by the R.H.S. of equation (2.68) and $|A(s)|$ is determinant of A(s).

We may observed that $|A(s)| = s\,D(s)$, where D(s) satisfies the following recurrence relation.

$$D_n(s) - (s+\psi_{N-n+1}+\phi_{N-n+1})\,D_{n-1}(s) + \psi_{N-n+1}\,\phi_{N-n+1}\,D_{n-2}(s) = 0; \quad 1 \le n \le N \quad ....(2.71)$$

With $D_{-1}(s) = 0$ and $D_0(s) = 1$

The zero's of $D_N(s)$ are the negatives of the eigen values of the matrix D(0), D(0) is a positive definite symmetric tri-diagonal matrix. Hence the roots of $D_N(s)$ are real negative and distinct.

Let $\alpha_1, \alpha_2, ..., \alpha_N$ be the roots of $D_N(s)$. Thus

$$P_z(n) = \frac{|A_{n+1}(s)|}{s \prod_{i=1}^{N} (s - \alpha_i)}; \qquad 0 \le n < N \quad .... (2.72)$$

Resolving the R.H.S. of $P_z(n)$ into partial fraction replacing s by (1-z)/z and comparing the coefficient of z^m, we have

$$P_m(n) = b_n + (i!/n!)\,(\mu)^{1-n}\,B_n \sum_{k=1}^{N} a_{kn}\,(1-\alpha_k)^m; \qquad 0 \le n \le i \quad ....(2.73)$$

$$P_m(n) = b_n + \sum_{k=1}^{n} a_{kn}\,(1-\alpha_k)^m; \qquad n = 1 \quad ....(2.74)$$

$$P_m(n) = b_n + \{(n-i)!/(N-n)!\} (\lambda/N)^{n-1} \prod_{j=1}^{n-1} (1-j\mu) \sum_{k=1}^{n-c_N} a_{kn} (1-\alpha_k)^m; \; i \le n \le c \quad \ldots\ldots (2.75)$$

$$P_m(n) = b_n + \{(n-i)!/(N-n)!\} (\lambda/N)^{n-1} \prod_{j=i}^{c-1} (1-j\mu)(1-c\mu) \sum_{k=1}^{n-c_N} a_{kn} (1-\alpha_k)^m; \; c \le n \le N \quad \ldots\ldots(2.76)$$

Where $B_n = \prod_{j=n}^{i-1} \{1-(N-j-1)\lambda\}$

For $c \le i \le N$

$$P_m(n) = b_n + \{(c-1)!/n!\} (\mu)^{c-n-1} (c\mu)^{i-c+1} B_n \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m; \quad \ldots\ldots (2.77)$$

$$P_m(n) = b_n + (c\mu)^{i-n} B_n \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m ; \; c<i \quad \ldots\ldots(2.78)$$

$$P_m(n) = b_n + \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m ; \quad n = i \quad \ldots\ldots(2.79)$$

$$P_m(n) = b_n + \{(n-i)!/(N-n)!\} (\lambda/N)^{n-i} \sum_{k=1}^{N} a_{kn} (1-\alpha_k)^m ; \quad i+1 \le n \le N \quad \ldots\ldots(2.80)$$

Where a_{kn}'s are defined as

$$\left.\begin{aligned} a_{kn} &= \frac{C_{N-1}(\alpha_k)\, D_n(\alpha_k)}{\alpha_k \prod_{j=1\neq k}^{N} (\alpha_k - \alpha_j)}, && \text{if } 0 \le n \le i \\ a_{kn} &= \frac{C_{N-1}(\alpha_k)\, D_1(\alpha_k)}{\alpha_k \prod_{j=1=k}^{N} (\alpha_k - \alpha_j)}, && \text{if } 0 \le n \le i \end{aligned}\right\} \quad \ldots\ldots (2.81)$$

With $C_n(s)$ and $D_n(s)$ being the determents obtained by the bottom right and top left square matrix formed from A(s) such that

$$|A(s)| = C_{N+1}(s) = D_{N+1}(s)$$

$C_n(s)$ and $Dn(s)$ may be determine by the following recurrence relations.

$$C_n(s) = (s+\phi_{N-n+1}+\psi_{N-n+1})\,C_{n-1}(s) + \psi_{N-n+1}\,\phi_{N-n+1}\,C_{n-2}(s) = 0;\quad 1 \le n \le N+1 \quad \text{....... (2.82)}$$

$$D_n(s) = (s+\phi_{n-1}+\psi_{n-1})\,D_{n-1}(s) - \psi_{n-2}\,\phi_{n-1}\,D_{n-2}(s)\,;\quad 1 \le n \le N+1 \quad \text{....... (2.83)}$$

With $D_0(s) = 1$ and $D_{-1}(s) = 0$

and

$$b_0^{-1} = \begin{cases} 1+\sum_{r=1}^{N} {}^{N}C_r\,(\lambda/N\mu)^r \dfrac{\prod_{i=1}^{r}\{1-(i-1)\,\mu\}}{\prod_{i=1}^{r}\{1-(N-i)\,\lambda\}} \\[2ex] 1+\sum_{r=1}^{c} {}^{N}C_r\,(\lambda/N\mu)^r \dfrac{\prod_{i=1}^{r}\{1-(i-1)\,\mu\}}{\prod_{i=1}^{r}\{1-(N-i)\,\lambda\}} + \sum_{i=c+1}^{c} \dfrac{N!}{(N-r)!c!}(\lambda/N)^r \end{cases}$$

$$\{(1-c\mu/c\mu)^{r-c}\}\,(1/(\mu)^c)\,\frac{\prod_{i=1}^{c}\{1-(i-1)\,\mu\}}{\prod_{i=1}^{r}\{1-(N-i)\lambda\}} \quad \text{.......(2.84)}$$

$$
b_r b_0^{-1} = \begin{cases} {}^{N}C_r\,(\lambda/N\mu)^r \dfrac{\prod_{i=1}^{r} \{1-(i-1)\,\mu\}}{\prod_{i=1}^{r} \{1-(N-i)\,\lambda\}} & ; \quad i \le r \le c \\[2ex] \{N!/(N-r)!c!\}\,(\lambda/N)^r\,(1-c\mu/c\mu)^{r-c}\,\{1/(\mu)^c\} \dfrac{\prod_{i=1}^{r} \{1-(i-1)\,\lambda\}}{\prod_{i=1}^{r} \{1-(N-i)\,\lambda\}} & ; \quad c+1 \le r \le N \end{cases} \quad \text{.......(2.85)}
$$

It may be noted that the probabilities $P_m(n)$ to remain bounded $|1+\alpha_k| < 1$ which is true if $(\psi_{k-1} + \phi_k) < 1$, under this condition the sum of the absolute values of the elements of the matrix $D(0) < 2$. Hence from Gerscehgorin's Theorem $|\alpha_k| < 2$. Hunder (1983) which implies that $|1+\alpha_k| < 1$.

Using IMSL package we can get eigen value of D(0). Since $(1+\alpha_k)^m$ as $m\to\infty$ the steady state distribution $P_m(n)$ may be defined as

$$
P(n) = \lim_{m\to\infty} P_m(n) = b_n; \qquad 0 \le n \le N
$$

Further we observe that the values of $P_m(n)$ have been expressed as sum of two expressions namely steady state and transient.

GENERAL CASE :

So for we have assumed that initial queue size is fixed and equal to i that is 1/z occurs only one position of the initial probabilities vector. This assumption is important when we are interested in the transient solution. The steady state solution does not depend on the initial probabilities vector.

We now consider a more general case of this problem, when there are more than one non zero element in the initial probabilities vector. The probabilities $Q_m(n)$, $\{n=0,1,\ldots.N\}$ defined as the probability of n customers in the system at epoch m irrespective of the state of system may be defined as

$$Q_m(n) = \sum_{i=1}^{N} P_m(n,1)\, P_0(i); \qquad 0 \le n \le N \qquad \ldots\ldots.(2.86)$$

Where $P_0(i)$ is the i^{th} element of the initial probability vector and $P_m(n,i)$ is the probability of the n machine in the system at epoch m assuming i as the initial number of machine.

CONTINUOUS TIME CASE OF GEOM(N)/C/N DISCRETE MODEL :

Letting $\lambda_n \quad = \quad \lambda_n\, \theta + 0(\theta)$

$\mu_n \quad = \quad \mu_n\, \theta + 0\,(\theta),$

and $m \quad = \quad t, \quad m+1 = t = \theta$

Substitute in equation (2.48) to (2.53), the difference equation in m can be transform to differential equations, in t, we can proceed as above to get continuous time solution from the transformed equations.

Alternatively the root of the equation can be changed to get the continous time solution from the final discrete time solution, the roots a_k of the $|A(s)| = 0$ are transform into. It is easy to see that $(1+a_{k\theta})^m \rightarrow$subintervals each of length such as $t = m\theta$.

Moreover parameter s may be treated as transform parameter in continuous time, the right hand side of eq. (2.58) will have one in the i^{th} place of instead of 1/z.

Important Performance Measures of Geom(N)/Geom(N)/C/N machine interference Discrete Model :

Using explicit expressions for $P_m(n)$, some important measures can be defined as under (for fixedi)

1. Mean number of customer in the system at epoch m

$$E(X_m) = \sum_{n=1}^{N} n\, P_m(n)$$

2. Mean number of customer in the queue(excluding those in service) at epoch m

$$E(Y_m) = \sum_{n=1}^{N} (n-c)\, P_m(n)$$

3. Probabilities that there are r or more customer in the system at epoch m is

$$\sum_{n=1}^{N} P_m(n)$$

4. Probability all servers are busy at epoch m

$$E(Z_m) = \sum_{n=1}^{N} P_m(n)$$

5. Relaxation Time (RT) [a measure of the length of time required by the system to settle to its stady-state Morese (1958)] may be defined as.

$$RT = \frac{1}{\min_{i=1}^{n} (-a_i)} \qquad \text{if } m>RT \text{ then } P_m(n) \sim P(n)$$

6. The Probability of balking at epoch m

$$E(A_m) = \sum_{n=0}^{N} (n/N)\, P_m(n)$$

7. The Probability of waiting upto epoch m in the queue by those joining is

$$E(B_m) = \sum_{n=c+1}^{N} (1-n/N)\, P_m(n)$$

NUMERICAL RESULTS :

We give below the numerical results for both the discrete and continuous cases for the model discussed above

Discrete Case :

Assume $c = 5, N = 20, \lambda = 0.04, \mu = 0.1, m = 10$

$\lambda = (N\text{-}i), \mu_i = i\mu$ for $0 \le i \le c$

$\mu_i = c\mu;\ c \le i \le N$

and $P_0(i) = 1/21$

Table 1 gives the probabilities $P_m(n)$ for $0 \le i \le 20$, the unconditional probabilities $Q_m(n)$ and the steady-state probabilities $P(n)$. The last five rows give the values for $E(X_m)$, $E(Y_m)$, $E(A_m)$ and $E(B_m)$.

The epoch to reach stady-state is m 220 is $\geq$ RT = 22.

Continuous Case :

Assume t = 10 and rest of the parameters as in Table 1 and Table 2 gives the probabilities $P_1(n)$ for different $i(0 \le i \le 20)$ the unconditional probabilities $Q_1(n)$ and the steady-state probabilities $P(n)$.

The time to reach steady-state is $t \cong 130$ which is >> RT = 20.

Finally the accuracy for the eigen values or roots that needed in such types of problems is very very large because the recurrence relations involved in computing the probabilities.

Table 1. : Probabilities $P_m(n)$, $Q_m(n)$ and $P(n)$ and important performance measure for symmetric and Geom(n)/C/N machine interference problem with C=5, M=30 and $\mu = 0.1$, $P_0(i) = 1/2!$; i = 0,1,................20

	0	1	2	...	18	19	20	$Q_m(n)$	P(n)
1	0.0005	0.0005	0.0005	...	0.0000	0.0000	0.0000	0.0002	0.0002
2	0.0059	0.0056	0.0053	...	0.0001	0.0000	0.0000	0.0022	0.0027
3	0.0328	0.0314	0.299	...	0.0006	0.0004	0.0002	0.0130	0.0161
4	0.0990	0.0954	0.0915	...	0.0031	0.0020	0.0013	0.0423	0.0532
5	0.1689	0.1642	0.1589	...	0.0099	0.0069	0.0047	0.0801	0.1021
6	0.1939	0.1909	0.1874	...	0.245	0.0184	0.0135	0.1088	0.1392
7	0.1828	0.1827	0.1824	...	0.0516	0.0415	0.0326	0.1299	0.1624
8	0.1424	0.1450	0.1477	...	0.0910	0.0779	0.0653	0.1387	0.624
9	0.0920	0.0957	0.0997	...	0.1342	0.1221	0.1089	0.1341	0.1392
10	0.0492	0.0524	0.0561	...	0.1657	0.1597	0.1512	0.1176	0.1021
11	0.0217	0.0238	0.0262	...	0.1706	0.1740	0.1746	0.0932	0.0638
12	0.0079	0.0089	0.0102	...	0.1461	0.1573	0.01669	0.0656	0.0339
13	0.0024	0.0027	0.0032	...	0.1031	0.1170	0.1311	0.0403	0.0150
14	0.0005	0.0007	0.0008	...	0.593	0.0709	0.0837	0.0209	0.0055
15	0.0001	0.0001	0.0001	...	0.272	0.0344	0.0427	0.0090	0.0017
16	0.0000	0.0000	0.0000	...	0.0097	0.0130	0.0170	0.0031	0.0004
17	0.0000	0.0000	0.0000	...	0.0027	0.0037	0.0051	0.0008	0.0001
18	0.0000	0.0000	0.0000	...	0.0001	0.0001	0.0001	0.0001	0.0001
19	0.0000	0.0000	0.0000	...	0.0001	0.0001	0.0001	0.0000	0.0000
20	0.0000	0.0000	0.0000	...	0.0000	0.0000	0.0000	0.0000	0.0000
Total	1.000	1.000	1.000	...	1.000	1.000	1.000	1.000	1.000
$E(X_m)$	6.6264	6.6931	6.7685	...	10.5867	10.8801	11.1736	8.4978	7.7358
$E(Y_m)$	1.8618	1.8702	1.9377	...	5.5912	5.8829	6.1753	3.5737	2.8301
$E(Z_m)$	0.8618	0.8671	0.8728	...	09963	0.9976	0.9985	0.9422	0.9278
$E(A_m)$	0.3313	0.3347	0.3384	...	0.5293	0.5540	0.5587	0.4249	0.3868
$E(B_m)$	0.4291	0.4337	0.4385	...	0.4603	0.4489	0.4366	0.4679	0.4778

Table 2. : Probabilities $P_t(n)$, $Q(n)$ and $P(n)$ and important performance measure for symmetric and M/M/C/N machine interference problem with t=10, N=20, m = 10 and $\lambda = 00.04$, $\mu = 0.1$, $P_0(i) = 1/2!$; i = 0,1,...............20

	0	1	2	...	18	19	20	$Q_m(n)$	P(n)
0	0.0078	0.0054	0.0037	...	0.0000	0.0000	0.0000	0.0011	0.0006
1	0.0429	0.0325	0.0243	...	0.0000	0.0000	0.0000	0.0073	0.0044
2	0.1115	0.0923	0.0750	...	0.0000	0.0000	0.0000	0.0228	0.0167
3	0.1820	0.01634	0.1434	...	0.0000	0.0000	0.0000	0.0456	0.0402
4	0.2080	0.2008	0.1892	...	0.0000	0.0000	0.0000	0.0657	0.0683
5	0.1736	0.1793	0.1805	...	0.0002	0.0001	0.0000	0.0720	0.0874
6	0.1247	0.1381	0.1493	...	0.0005	0.0002	0.0001	0.746	0.1049
7	0.0774	0.0920	0.1071	...	0.0015	0.0006	0.0003	0.0747	0.1175
8	0.0415	0.0530	0.0664	...	0.0040	0.0019	0.0008	0.0737	0.1222
9	0.0193	0.0262	0.0356	...	0.0099	0.0050	0.0024	0.0725	0.1173
10	0.0077	0.0112	0.0163	...	0.0225	0.0124	0.0063	0.0715	0.1032
11	0.0026	0.0041	0.0064	...	0.0457	0.0275	0.0153	0.0706	0.0826
12	0.0008	0.0013	0.0021	...	0.0820	0.0546	0.0335	0.0693	0.0594
13	0.0002	0.0003	0.0006	...	0.1281	0.0954	0.0650	0.0669	0.0380
14	0.0000	0.0000	0.0001	...	0.1706	0.1441	0.1104	0.0624	0.0130
15	0.0000	0.0000	0.0000	...	0.1887	0.1837	0.1607	0.0544	0.0102
16	0.0000	0.0000	0.0000	...	0.1668	0.1912	0.1950	0.0430	0.0041
17	0.0000	0.0000	0.0000	...	0.1115	0.1548	0.1893	0.0291	0.0013
18	0.0000	0.0000	0.0000	...	0.0518	0.0902	0.1378	0.0157	0.0003
19	0.0000	0.0000	0.0000	...	0.0144	0.0329	0.0669	0.0059	0.0001
20	0.0000	0.0000	0.0000	...	0.0018	0.0054	0.0162	0.0012	0.0000
Total	1.0000	1.0000	10.000	...	1.0000	1.0000	1.0000	1.0000	1.0000
$E(X_m)$	4.4244	4.7178	5.0294	...	14.5384	15.2087	15.8790	9.5082	8.0480
$E(Y_m)$	0.5419	0.6793	0.8460	...	9.5384	10.2087	15.7890	9.5082	0.0480
$E(Z_m)$	0.4477	0.5056	0.5644	...	0.9999	1.0000	1.0000	0.8575	0.8698
$E(A_m)$	0.2212	0.2359	0.2515	...	0.7369	0.7604	0.7940	0.4754	0.4024
$E(B_m)$	0.1784	0.2108	0.2456		0.2729	0.2395	0.2060	0.3507	0.4235

CHAPTER - 3

DISCRETE TIME TRANSIENT ANALYSIS OF A SYSTEM WITH QUEUE DEPENDENT SERVERS

CHAPTER - THREE

DISCRETE TIME TRANSIENT ANALYSIS OF A SYSTEM WITH QUEUE DEPENDENT SERVERS

This chapter of thesis deals with diserete time analysis of an infinite capacity Markovian Queueing model with number of heterogeneous channels depending on system size. The model consists of one regular channel and a maximum of C-additional channel by defining probability generating functions.

Introduction :

Most of the Queueing theary literature concentrates on finding the steady state solutions or approximations. Very little seems to have been done to evaluate the transient solutions. Even at time steady state solutions are difficult to compute.

Chaudhry, Agarwal & Templetion (1992) have mostly concentrated on this using the techniques of roots. Earlier attempts at finding transient solutions can be attributed to Tacka's (1962) & Morse (1959). However there are computations difficulties with their methods. Now with the increased skill available in computations with the use of computers researches especially in computer science have started looking for transient solutions and easy to compute closed form solutions. Sharma & Dass (1988) have provided transient solutions to a class of Markovian Queueing models in Queueing Theory. However they did not concentrate on the computational difficulties of finding roots or eigen values if the matrices involved are large.

Most application of Queueing theory involves queues which are emptied and restarted periodically. These queues never reach equilibrium state. Hence steady state solutions are not always adequate and it is desirable to have time dependent solutions.

Sharma and Shobha (1988) have considered multiserver Markovian Queueing model with finite waiting space and derived transient results.

Sharma and Tarabia (2000) have obtained the transient state probabilities of a Singh server Markovian Queue with finite source in a closed form using Laplace transform technique. Tarabia (2001) has analysed the transient behaviour of a non-empty M/M/1/N queueies model using Laplace transform technique.

In real situation it is common that the executive concerned is obliged to offer special service facility to his customers to gain their good will. Systems with additional servers may be required in places like banks, tax offices, transport facilities etc. hence a system with queue depended servers is proposed, Gross and Harries (1985) have mentioned that the transient solution of single channel queue becomes more complicated with the restriction on waiting room capacity is relaxed. However in this chapter we presents the discrete time transient solution of an finite capacity queuing model with additional servers. Here we assume that the system capacity is N.

Moreover, no attempt seems to have been made to obtain similar results in discrete time for finite waiting space problems in Queueing theory.

As the transient solution is not independent of the initial state of the system's behaviors, further, some systems may not exist long enough, to reach their steady state.

There are several systems which operate at discrete times see Kobayashi (1983). As a result it becomes important to study them, in such cases events are clock controlled.

In this chapter we analyze a discrete time transient analysis of a Markovian Queueing model of a system with queue dependent servers. Such problems occur not only in Queueing theory but also in bio- science and now in computer science we given closed

form solution to this class of problems in terms of the roots of a polynomial in Z- transform and results are computed even where the matrices involved are large. It is also shown how the results for the continuous can be obtained. Interesting analogy exists between the discrete time models and their continuous time counter parts.

Results presented in this chapter further unify the treatment given by Chaudhry, Kapur, Templeton (1991). It is worth noting though, continuous time models are particular cases of discrete time models, yet this area of research has remain neglected. Assume that the customers arrive individually at a service facility in accordance with a Poisson process having rate λ. The queue discipline is FIFO. The service facility consists of one regular service channels. The system starts with a regular service channels. The system starts with a regular service channel and c- additional service channel having service rate μ.

The regular channel is always open irrespective of queue length. If the system is empty then the regular channel is idle. If the number of customers in the system increases to r (0) =1 then the regular channel will become busy. If the number of customer in the system is r (1) then one additional service channel with service rate $\mu^{(1)}$ is started, which will be dropped at the termination of a service, if the system size becomes less than r (1) at that epoch.

When two channels are operating, if the number of units in the system increases to r (2) (> r(1)), the second additional channels is limited to a maximum of c and when all the c- channels are operating together with the regular channels.

The queue is allowed to grow without limit initially system is assumed to have c- units. Let $p_m(n)$ be the probability of n customers in the system at m^{th} epoch or time slot.

Let $\mu^i = \mu + \sum_{n=1}^{i} \mu^{(n)}$

The following difference equation may easily be written as

$$p_{m+1}(0) - p_m(0) = -\lambda p_m(0) + \mu p_m(1) \quad \text{.......(3.1)}$$

$$p_{m+1}(n) - p_m(n) = -(\lambda+\mu_k)\ p_m(n) + p_m(n-1)\lambda + \mu_k\ p_m(n+1) \quad \text{.......(3.2)}$$

$$; r(k) \le n \le r(k+1)$$

$$p_{m+1}(r(j)-1) - p_m(r(j)-1) = -(\lambda+\mu_j-1)\ p_m(r(j)-1) + \mu_j\ p_m(r(j)) + \lambda\ p_m(r(j)-2)$$

$$\text{Where } j = 1,2,3 \quad \text{.........(3.3)}$$

$$p_{m+1}(n) - p_m(n) = -\lambda\ p_m(n-1) + \mu_c\ p_m(n+1) - (\lambda+\mu_c)\ p_m(n) \quad \text{.......(3.4)}$$

$$; n \ge r(c)$$

Defining the probability generating function as

$$P(z,m,n) = \sum_{n=0}^{r(c)} \sum_{m=0}^{\infty} p_m(n)\, z^m + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} p_m(n+r(c))\, z^m \qquad |z| \le 1$$

$$= \sum_{m=0}^{\infty} z^m \sum_{m=0}^{r(c)} p_m(n) \sum_{m=0}^{\infty} z^m R(m) + \sum_{m=0}^{\infty} z^m \sum_{n=1}^{\infty} p_m(n+r-1)$$

using the initial condition, we have

$$P(z,0) = P_0(n) = z^{K(i)} \text{ or } z^{\delta(i)}$$

now taking the probability generating function of equation (3.1), (3.2), (3.3), (3.4)

$$R(m) = \sum_{n=0}^{r(c)} p_m(n)$$

$$P(z,m.n) = R(m) + \sum_{n=1}^{\infty} p_m(n+r(c))\, z^n$$

$$P(z,m,n) = \sum_{m=0}^{\infty} \sum_{n=0}^{r(c)} p_m(n) + \sum_{m=0}^{\infty} \sum_{n=1}^{\infty} p_m(n+r(c))\, z^n$$

$$P(z, m, n) = \sum_{m=0}^{\infty} \sum_{n=1}^{N} z^m p_m(n)$$

$$= \sum_{n=1}^{N} z^0 p_0(n) + z^1 p_1(n) + z^2 p_2(n) + \ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$p_m(n)$ = Coeff of z^m in the expansion of P (z, m, n).

Multiplying eq. (3.1) by z^m and summing from 0 to ∞, We have

$$z^m p_{m+1}(0) - p_m(0) z^m = -\lambda Pm(10) z^m + \mu Pm(1). z^m$$

$$\sum_{m=0}^{\infty} z^m p_{m+1}(0) - p_m \sum_{m=0}^{\infty}(0) z^m = -\lambda \; p_m(0) \sum_{m=0}^{\infty} z^m + \mu p_m(1) . z^m$$

$$(1/z) \sum_{m=0}^{\infty} z^{m+1} p_{m+1}(0) - p(z, m, o) = -\lambda p(z, m, o) + \mu p(z, m, 1)$$

$$(1/z)[p(z, m, o) - p_0(0)] - P(z, m, 0) = -\lambda P(z, m, o) + \mu P(z, m, 1)$$

$$(1 - z)/z\, P(z, m, o) - 1/2\, p_0(0) = -\lambda P(z, m, o) + \mu P(z, m, 1)$$

Denoting $(1 - z)/z = s$, we have

$$s P(z, m, o) - 1/z\, p_0(0) = -\lambda P(z, m, o) + \mu P(z, m, 1)$$

$$(s+\lambda) P(z, m, o) - \mu P(z, m, 1) = 1/z\, p_0(0) \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3.5)$$

Multiplying eqn (3.2) by z^m and summing over 0 to ∞, we have

$$\sum_{m=0}^{\infty} p_{m+1}(n) z^m - \sum_{m=0}^{\infty} p_m(n) z^m = -(\lambda+\mu_k) \sum_{m=0}^{\infty} P_m(n) z^n + 1 \sum_{m=0}^{\infty} p_m(n-1) z^m + \mu_R \; p_m(n+1) \; z^m$$

$$(1/z) \sum_{m=0}^{\infty} p_{m+1}(n) z^m = 1 - \sum_{m=0}^{\infty} p_m(n) z^m = -(\lambda+\mu_k) \sum p_m(n) z^m + \lambda \sum_{m=0}^{\infty} p_m(n-1) z^m + \mu_k \sum_{m=0}^{\infty} p_m(n+1) z^m$$

$$(1/z)[P(z, m, o) - P_0(n)] - P(z, m, n) = -(\lambda+\mu_k) P(z, m, n) + P(z, m, n-1) + \mu_\kappa P(z, m, n+1)$$

$$(1-z)/z\, P(z, m, n) - 1/z\, p_0(n) = -(\lambda+\mu_k)\, P(z, m, n) + \lambda\, P(z, m, n-1) + \mu_k\, P(z, m, n+1)$$

$$s\, P(z, m, n) + P(z, m, n) - \lambda\, P(z, m, n-1) - \mu_\kappa\, P(z, m, n+1) = 1/z\, P_0(n)$$

$$(s + \lambda + \mu_r)\, P(z, m, n) - \lambda\, P(z, m, n-1) - \mu_r\, P(z, m, n+1) = 1/z\, P_0(n)\ ;\ r(r) \le n \le r(r+1) \quad \text{............(3.6)}$$

Multiplying eq^n (3.3) by z^m and summing from 0 to ∞, we have

$$\sum_{m=0}^{\infty} p_{m+1}(r(j)-1)\, z^m - \sum_{m=0}^{\infty} p_m(r(j)-1)\, z^m = -(\lambda+\mu_j) \sum_{m=0}^{\infty} p(r(j)-1)\, z^m + \mu_j \sum_{m=0}^{\infty} p_m(r(j)) z^m$$
$$+ \lambda \sum_{m=0}^{\infty} p_m(r(j)-2)\, z^m$$

$$1/z \sum_{m=0}^{\infty} p_{m+1}(r(j)-1) z^{m+1} - \sum_{m=0}^{\infty} p_m(r(j)-1)\, z^m = -(\lambda + \mu_{j-1}) \sum_{m=0}^{\infty} p_m(r(j)-1)\, z^m +$$
$$\mu_j \sum_{m=0}^{\infty} p_m(r(j))\, z^{m+1} + \sum_{m=0}^{\infty} p_m(r(j)-2)\, z^m$$

$$1/z\, [P(z, m, (r(j)-1))\, P_0\, (r(j)-1)] - P(z, m, r(j)-1)$$
$$= -(\lambda + \mu_{j-1})\, P(z, m, r(j)-1) + \mu_j\, P(z, m, r(j)) + \lambda\, P(z, m\, r(j)-2)$$

$$(1-z)/z\, P(z, m, r(j)-1) - 1/z\, p_0\, (r(j)-1)$$
$$= -(\lambda + \mu_{j-1})\, P(z, m, r(j)-1) + \mu_j\, P(z, m, r(j)) + \lambda\, P(z, m\, r(j)-2)$$

$$s\, P(z, m, r(j)-1) + -(\lambda + \mu_{j-1})\, P(z, m, r(j)-1) - \mu_j\, P(z, m, r(j)) - \lambda\, P(z, m\, r(j)-2)$$
$$= 1/z\, p_0(r(j)-1)$$

$$(s+\lambda + \mu_j)\, P(z, m; r(j)-1) + \mu_j\, P(z, m, r(j)) + \lambda\, P(z, m, r(j)-2) = 1/z\, p_0(r(j)-1) \quad \text{.........(3.7)}$$

Multiplying eqn. (3.4) by z^m and summing over 0 to ∞, we get

$$\sum_{m=0}^{\infty} p_{m+1}(n)\, z^m - \sum_{m=0}^{\infty} p_m(n)\,.\, z^n = \lambda \sum_{m=0}^{\infty} p_m(n-1)\, z^m + \mu_c \sum_{m=0}^{\infty} p_m(n+1)\, z^m$$

$$- (\mu_k\, \lambda) + \sum_{m=0}^{\infty} P_m(n)\, Z^m$$

$$1/z \sum_{m=0}^{\infty} p_{m+1}(n)\, z^{m+1} - \sum_{m=0}^{\infty} p_m(n)\, z^n = \lambda \sum_{m=0}^{\infty} p_m(n-1)\, z^m + \mu_c \sum_{m=0}^{\infty} p_m(n+1)\, z^m$$

$$- (\mu_k\, \lambda) + \sum_{m=0}^{\infty} p_m(n)\, z^m$$

$$1/z\, [P(z, m, (n) - p_0(n)] - P(z, m, n) = -\lambda\, P(z, m, n-1) + \mu_c\, P(z, m, n+1) - (\lambda+\mu_c)\, P(z, m, n)$$

$$(1-z)/z\, P(z, m, (n) - p_0(n)] - 1/z\, p_0(n) = -\lambda\, P(z, m, n-1) + \mu_c\, P(z, m, n+1) - (\lambda+\mu_c)\, P(z,m,n)$$

$$s\, P(z, m, n) + (\lambda + \mu_c)\, P(z, m, n) + \lambda.\, P(z, m, n-1) - \mu_c\, P(z, m, n+1) = 1/z\, p_0(n)$$

$$(s+ \lambda + \mu_c)\, p(z, m, n) - \lambda\, p(z, m, n-1) + \mu_c\, p(z, m, n+1) = 1/z\, p_0(n)$$

$$-\lambda\, p(z, m, n-1) + (s+ \lambda + \mu_c)\, p(z, m, n) + \mu_c\, p(z, m, n+1) = 1/z\, p_0(n) \quad \ldots\ldots\ldots\ldots(3.8)$$

$$AP = [\delta_{ij}\ \ \delta_{i1} \ldots\ldots \delta_{iN}]'$$

Where A is a real tri-diagonal matrix of order (N+1)x (N+1) matrix.

P is a column vector.

δ_{ij} is the Kronecker delta defined as

$$\delta ij = \begin{cases} \tfrac{1}{2} & \text{if } j=i \\ 0 & \text{if } j \neq i \end{cases}$$

$$
\begin{array}{cc}
 & \begin{array}{ccccccccccccccccc} 0 & 1 & 2 & - & r(1)-1 & r(1) & r(1)+1 & - & r(2)-1 & r(2) & r(2)+1 & - & r(c)-3 & r(c)-2 & r(c)-1 & - & N \end{array} \\
\begin{array}{c} 0 \\ 1 \\ 2 \\ - \\ r(1)-1 \\ r(1) \\ r(1)+1 \\ r(2)-1 \\ r(1) \\ r(2)+1 \\ - \\ - \\ r(c)-2 \\ r(c)-1 \\ - \\ N \end{array} &
\left[\begin{array}{ccccccccccccccccc}
s+\lambda & -\mu & 0 & - & 0 & - & - & - & - & - & 0 & - & - & 0 & 0 & & \\
-\lambda & (s+\lambda+\mu) & -\mu & - & 0 & - & - & - & - & - & 0 & - & - & 0 & 0 & & \\
0 & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
0 & 0 & 0 & - & -\lambda & (s+\lambda+\mu) & - & - & - & -\mu & 0 & - & - & 0 & 0 & & \\
0 & 0 & 0 & - & - & -\lambda & - & - & - & (s+\lambda+\mu) & -\mu & - & - & 0 & 0 & & \\
0 & 0 & 0 & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & -\lambda\ (s+\lambda+\mu_{c-1}) & -\mu_{c-1} & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & & \\
- & - & - & - & - & - & - & - & - & - & - & - & - & - & - & &
\end{array}\right]
\end{array}
\left[\begin{array}{c}
P(z,m,0) \\ P(z,m,1) \\ P(z,m,2) \\ - \\ - \\ - \\ P(z,m,r(1) \\ P(z,m,r(1)+1) \\ - \\ - \\ P(z,m,r(2)) \\ P(z,m,r(2)+1) \\ P(z,m,r(c)-2 \\ P(z,m,r(c)-1) \\ - \\ P(z.m.n)
\end{array}\right]
=
\left[\begin{array}{c}
1/z \\ 1/z \\ 1/z \\ - \\ - \\ - \\ . \\ 1/z \\ - \\ - \\ 1/z \\ . \\ . \\ 1/z \\ 1/z \\ 1/z
\end{array}\right]
$$

CHAPTER - 4

A DISCRETE TIME TRANSIENT SOLUTION OF INTERDEPENDENT TANDEM QUEUEING MODEL WITH ARRIVAL IN BULK

CHAPTER – FOUR

A DISCRETE TIME TRANSIENT SOLUTION OF INTERDEPENDENT TANDEN QUEUEING MODEL WITH ARRIVAL IN BULK

ABSTRACT

In this Chapter we study a discrete time transient solution of an interdependent queueing model having three services in series. The customers arrive at the first node in groups of size k, the arrival of cutomers and number of service completion at each node are correlated, following correlated poisson process. Services are provided singly. For this queueing model, the average number of customers in the network, and the probability that the network is idle are derived. This model may be useful in finding the system performance measures for communication networks.

INDRODUCTION :

The communication system can be modelled as network of interconnected queues by taking the messages as customers communication buffer as waiting line and all activities necessary in transmission of the messages as services. In communication networks it is generally known that the packet switching gives better utilization over the circuit or massage switching and yields relatively short network delay. In packet switching the messages are divided in to small packets of random length each packet will have an independent header for routing. Due to the unproductive nature of demand, transmission lines congestion occurs in communication systems. This leads to model communication networks as a tandem queueing system. Several authors have studied the communication networks. Yakuo (1993) did throughout analysis of tandem queues, Jenq (1994) provided approximation for

packetized voice traffic in statistical multiplexer in process but most of them have considered the independent assumption on the service and arrival process. However, in some communication systems like, store and forward communication network standard type of independence assumption is realistically inappropriate due to the fact that message generally preserve their length as they transverse the network. The inter-arrival and service requirements at queues internal to the system are thus dependent as there are queueing processes at each of the nodes of the network through which the same messages are routed. These dependencies can have a marked effect on system performance and must be accounted for any realistic analysis.Recently, Srinivasa et. al. (2000) worked on interdependent tandem queueing model, who developed and analyzed an interdependent communication network which is composed of several nodes of transmission through direct dependents between arrivals and transmission. But, still there seems to be no work done on the interdependent tandem queueing models with bulk arrive, although with reference to communication networks, it is most natural with respect to actual operations of such systems.

It is in the reference to the communication networks, is assume that the massages are packetized at source stored in buffer for transmission. So for the concerned queueing model we have considered that customers are arriving in batches of size K. There are three nodes (servers) in series, after being served at the first node the customer leaves the network with probability p_1 or join the second buffer with probability p_1 after being transmitted to the second node the customer join the third buffer with probability $(1 - p_1)$ and leaves the system with probability $(1 - p_2)$.

Here we assume that the arrival of customers (Packetes) and number of transmission services at each node (server) are correlated and follow correlated poisson process having the joint probability mass functions of the form (parallel to Rao et.al. 2000).

Here we assume that the arrival of customers (Packetes) and number of transmission services at each node (server) are correlated and follow correlated poisson process having the joint probability mass functions of the form (parallel to Rao et.al. 2000).

$$P\,(X_1 = x_1, X_2, = x_2, X_3 = x_3, X_4 = x_4; m) = e^{-(k\lambda + \mu 1 + \mu 2 + \mu 3 + - 3\in)}m \sum_{j=0}^{\min (x_1, x_2, x_3, x_4)}$$

$$\frac{[k\lambda - \in)m\,]^{x_1 - j}\,[\mu_1 - \in)m\,]^{x_2 - j}\,[\mu_2 - \in)m\,]^{x_3 - j}\,[\mu_2 - \in)m\,]^{x_4 - j}\,(\in m)^j}{(x_1 - j)!\,(x_2 - j)!\,(x_3 - j)!\,(x_4 - j)!\,J!}$$

$$x_1, x_2, x_3, x_4 = 0, 1, 2, \ldots\ldots\ldots\ldots$$

$$0 < (\lambda, \mu, \varepsilon) < \min (\lambda, \mu_1),\ i = 1, 2, 3$$

Here

$k\lambda$ = mean arrival rate of packets

μ_1 = mean transmission rate of first node

μ_2 = mean transmission rate of second node

μ_3 = mean transmission rate of third node

$\in$ = mean dependence rate of the communication net work.

The schematic diagram of network is given,

Steady state difference differential Equations of Network :

Let $P_m\,(n_1, n_2, n_3)$ be the probability that there are n_1 packets in the first buffer, n_2 packets at the second buffer, n_3 packets at the third buffer at time t. Then the difference – differential equations of the communications network with three node are :

$P_{m+1}\,(n_1, n_2, n_3) - P_m\,(n_1, n_2, n_3) = -\,(\lambda + \mu_1 + \mu_2 + \mu_3 - 4\in)\,P_m\,(n_1, n_2, n_3) + (k\lambda - \in)$

$P_m\,(n_1 - k, n_2, n_3) + (\mu_1 - \in)\,p_1\,P_m(n_1+1, n_2, n_3) + Pm(n_1+1, n_2-1, n_3) + (\mu_2 - \in)\,p_1$

$P_m(n_1+1, n_2, n_3) + (\mu_1 - \in) p_1 (1 - p_2) P_m(n_1, n_2+1, n_3-1) + (\mu_2 - \in) p_2 + P_m(n_1+1, n_2, n_3+1)$

for $n_1, n_2, n_3 > 0$ ....(4.1)

$P_{m+1} (0, n_2, n_3) - P_m (0, n_2, n_3) = - (\lambda + \mu_2 + \mu_3 - 3\in) P_m (0, n_2, n_3) + (\mu_1 - \in) p_1$

$P_m(1, n_2, n_3) + (\mu_2 - \in) (1 - p_2) P_m(n_1+1, n_2-1, n_3) + (\mu_2 - \in) p_2 P_m(0, n_2+1, n_3) +$

$(\mu_2 - \in) (1 - p_2) P_m(1, n_2+1, n_3-1) (\mu_3 - \in) P_m(0, n_2, n_3+1) +$ for $0, n_2, n_3 > 0$..........(4.2)

$P_{m+1} (n_1, 0, n_3) - P_m (n_1, 0, n_3) = - (\lambda + \mu_1 + \mu_3 - 3\in) P_m (n_1, 0, n_3) + (k\lambda - \in) P_m(n_1-k, 0, n_3) +$

$(\mu_1 - \in) p_1 P_m(n_1+1, 0, n_3) + (\mu_2 - \in) p_2 P_m(n_1, 1, n_3) + (\mu_2 - \in) (1 - p_2) P_m(n_1, 1, n_3-1) + (\mu_2 -$

$\in) (1 - p_3) P_m(1, 0, n_3 -1)$ for $n_1, n_2 > 0, n_2 = 0$ ....(4.3)

$P_{m+1} (n_1, n_2, 0) - P_m (n_1, n_2, 0) = - (\lambda + \mu_1 + \mu_2 -3\in) P_m(n_1,n_2,0) + (k\lambda -\in) P_m (n_1-k, n_2, 0)$

$(\mu_1 - \in) p_1 P_m(n_1+1, n_2, 0) + (\mu_1 - \in) (1 - p_1) P_m(n_1+1, n_2, 0) + (\mu_2 - \in) p_2 P_m(n_1, n_2+1, 0) +$

$(\mu_3 - \in) (1 - p_2) P_m(n_1, n_2, 1)$ for $n_1, n_2, > 0, n_3 = 0$ ....(4.4)

$P_{m+1} (0, 0, n_3) - P_m (0, 0, n_3) = - (\lambda + \mu_3 -2\in) P_m(0, 0, n_3) + (\mu_1 - \in) p_1 P_m(1, 0, n_3) + (\mu_2 - \in)$

$p_2 P_m(0, 1, n_3) + (\mu_2 - \in) (1 - p_2) P_m(0, 1, n_3-1) + (\mu_3 - \in) P_m(0, 0, n_3+1)$

for $n_1 = n_2 = 0, n_3 > 0$ ....(4.5)

$P_{m+1} (n_1, 0, 0) - P_m (n_1, 0, 0) = - (\lambda + \mu_1 -2\in) P_m(n_1, 0, 0) + (k\lambda -\in) P_m (n_1-k, 0, 0)$

$(\mu_1 - \in) p_1 P_m(n_1+1, 0, 0) + (\mu_2 - \in) p_2 P_m(n_1, 1, 0) + (\mu_2 - \in) (1 - p_2) P_m(n_1, 1, 0) +$

$(\mu_3 - \in) P_m(n_1, 0, 1)$ for $n_2, n_3, = 0, n_1 > 0$ ....(4.6)

$P_{m+1} (0, n_2, 0) - P_m (0, n_2, 0) = - (\lambda + \mu_2 -2\in) P_m(0, n_2, 0) + (\mu_1 - \in) p_1 P_m(1, n_2, 0) +$

$(\mu_1 - \in) (1 - p_2) P_m(0, n_2 -1, 0) + (\mu_2 - \in) p_2 P_m(0, n_2+ 1, 0) + (\mu_2 - \in) P_m(0, n_2, 1)$

for $n_1 = n_3, = 0, n_2 > 0$ ....(4.7)

$P_{m+1} (0, 0, 0) - P_m (0, 0, 0) = - (\lambda - \in) P_m(0, 0, 0) + (\mu_1 - \in) P_m(1, 0, 0) + (\mu_2 - \in) p_2$

$P_m(0, 1, 0) + (\mu_3 - \in) P_m(0, 0, 1)$ for $n_1 = n_2 = n_3 > 0$ ....(4.8)

Assuming the system is in steady state and using the boundary conditions and solving iteratively, we obtain the joint probability mass function of the contents in three buffers after reaching steady state as :

$$P_m(n_1, n_2, n_3) \quad = \quad \frac{(\lambda - \in)^{n_1+n_2+n_3} (1 - p)^{n_2+n_3} (1 - p)^{n_3}}{(\mu_1 - \in)^{n_1} (\mu_2 - \in)^{n_2} (\mu_3 - \in)^{n_3}}$$

$$\frac{(\mu_1 - k\lambda) [(\mu_1 - p_1 \in -k\lambda (1 - p)] [\mu_3 + \{ p_1 . p_2 - (p_1 + p_2 \}.\varepsilon - k\lambda (1 - p_1) (1 - p_2)]}{(\mu_1 - \varepsilon) (\mu_2 - \varepsilon) (\mu_3 - \varepsilon)}$$

$$\text{for } (\lambda - \varepsilon) < \text{minimum}[(\mu_1 - \varepsilon), \frac{(\mu_2 - \varepsilon)}{(1 - p_1)}, \frac{(\mu_2 - \varepsilon)}{(1 - p_1)}] \qquad \text{...........(4.9)}$$

From this equation we can analyze the system performance measures by obtaining the system characteristics.

The probability that the network is idle is :

$$P_m(0, 0, 0) \quad =$$

$$\frac{(\mu_1 - k\lambda) [(\mu_2 - p_1 \in -k\lambda (1 - p_1)] [\mu_3 + \{ p_1 . p_2 - (p_1 + p_2 \}.\varepsilon - k\lambda (1 - p_1) (1 - p_2)]}{(\mu_1 - \varepsilon) (\mu_2 - \varepsilon) (\mu_3 - \varepsilon)} \qquad \text{....................(4.10)}$$

From this equation we see that probability of the idleness of the network increases when the mean dependent rate increases for fixed value of the other parameters :

The joint probability of having n_1 packets in the first buffer and n_2 packets in the second buffer is

$$P_m(n_1, n_2,) \quad = \quad \frac{(\lambda - \in)^{n_1+n_2} (1 - p)^{n_2}}{(\mu_1 - \in)^{n_1} (\mu_2 - \in)^{n_2}}$$

$$\frac{(\mu_1 - k\lambda) [(\mu_1 - p_1 \in -k\lambda (1 - p_1)]}{(\mu_1 - \varepsilon) (\mu_2 - \varepsilon)} \qquad \text{.............(4.11)}$$

The joint probability of having n_2 packets in the second buffer and n_3 packets in the third buffer is :

$$P_m(n_2, n_3) = \frac{(\lambda - \in)^{n_2+n_3} (1 - p_1)^{n_2+n_3} (1 - p)^{n_3}}{(\mu_2 - \in)^{n_2} (\mu_3 - \in)^{n_3}}$$

$$\frac{(\mu_1 - k\lambda) [(\mu_2 - p_1\in -k\lambda (1 - p_1)] [\mu_3 + \{ p_1 . p_2 - (p_1 + p_2\}.\varepsilon - k\lambda (1 - p_1) (1 - p_2)]}{(\mu_1 - \varepsilon) (\mu_2 - \varepsilon) (\mu_3 - \varepsilon)}$$

....................(4.12)

The joint probability of having x_1 packets in the second buffer and x_3 packets in the third buffer is :

$$P_m(n_1, n_2, n_3) = \frac{(\lambda - \in)^{n_1+n_3} (1 - p_1)^{n_3} (1 - \pi_2)^{n_3}}{(\mu_1 - \in)^{n_1} (\mu_3 - \in)^{n_3}}$$

$$\frac{(\mu_1 - k\lambda) [(\mu_3 +\{ p_1 . p_2 - (p_1 + p_2\}.\varepsilon - k\lambda (1 - p_1) (1 - p_2)]}{(\mu_1 - \varepsilon) (\mu_3 - \varepsilon)}$$

..................(4.13)

The average number of packets in the network is :

$$L = \frac{\begin{aligned}&(\lambda_1 - \varepsilon) [\{\mu_2 - p_1\in -k\lambda (1 - p_1) \{\mu_3 + \varepsilon\{ p_1 . p_2 - (p_1 + p_2\} - k\lambda (1 - p_1) (1 - p_2)\} \\ &+ (1 - p_1) (\mu_1 - k\lambda) \{\mu_3 + \varepsilon\{ p_1 . p_2 - (p_1 + p_2\} - k\lambda (1 - p_1) (1 - p_2)\} \\ &+ (1 - p_1) (1 - p_2) (\mu_1 - k\lambda) \{\mu_2 - k\lambda (1 - p_1)\}]\end{aligned}}{(\mu_1 - k\lambda) \{\mu_2 - p_1\in -k\lambda (1 - p) \{\mu_3 + \varepsilon\{ p_1 . p_2 - (p_1 + p_2\} - k\lambda (1 - p_1) (1 - p_2)\}}$$

From this equation we observe that the mean number of packets in the network is a an increasing function of λ. The average buffer content of the network is a decreasing function of mean transmission rates at first, second and third nodes when other parameters are fixed. The average buffer content of the network is a decreasing function of the mean dependence rate for fixed values of the other parameters. So by regulating the dependence parameter, one

can increase or decrease the mean number of packets in all the buffers and in each intermediate buffer. The joining probabilities of the network $1 - p_1$ and $1 - p_2$ have an influence on the mean number of packets in the buffers. By decreasing $1 - p_1$ and $1 - p_2$ one can reduce the conquestion in the intermediate queues.

The mean number of packets in first buffer is :

$$Lq_1 = \frac{(\lambda - \in)^2}{(\mu_1 - \in)(\mu_1 - \in)}$$

Then mean number of packets in second buffer is :

$$Lq_2 = \frac{(\lambda - \in)^2 (1 - p_1)^2}{(\mu_1 - \in)(\mu_1 - p_1 \varepsilon - k\lambda(1 - p_1))}$$

Then mean number of packets in third buffer is :

$$Lq_2 = \frac{(\lambda - \in)^2 (1 - p_2)^2}{(\mu_3 - \in)[\mu_3 + \varepsilon\{p_1 . p_2 - (p_1 + p_2)\} - k\lambda (1 - p_1)]}$$

The through put of the first transmitter is :

$$\mu_1(1 - P_0\ldots..) = \frac{\mu_1(\lambda - \in)}{\mu_1(\mu_1 - k\lambda)}$$

The average delay in the first buffer is :

$$\frac{L_1}{\mu_1(1 - P_0)} \quad \frac{(\mu_1 - \in)}{\mu_1(\mu_1 - k\lambda)}$$

Through put of the second transmitter is :

$$\mu_2(1 - P_0\ldots..) = \frac{\mu_2(\lambda - \in)(1 - p_1)}{(\mu_2 - \in)}$$

The average delay in the second buffer is :

$$\frac{L_2}{\mu_2(1-P_0)} = \frac{(\mu_2 - \in)}{\mu_2\,[\mu_2 - p_1\varepsilon - k\lambda\,(1-\ p_1)]}$$

The through put of the third transmitter is :

$$\mu_3(1-P_0\ldots..) = \frac{\mu_3(\lambda - \in)\,(1-p_1)\,(1-p_2)}{(\mu_3 - \in)}$$

The average delay in the third buffer is :

$$\frac{L_3}{\mu_3(1-P_0\ldots)} = \frac{(\mu_3 - \in)}{\mu_3\,[\mu_3 + \varepsilon\{p_1.p_2 - (p_1+p_2)\} - k\lambda\,(1-p_1)]}$$

CHAPTER - 5

A DISCRETE TIME TRANSIENT ANALYSIS OF A PRIORITY TANDEM QUEUE WITH ARRIVAL IN BATCHES AND HAVING NO INTERMEDIATE BUFFER

CHAPTER – FIVE

A DISCRETE TIME TRANSIENT ANALYSIS OF A PRIORITY TANDEM QUEUE WITH ARRIVAL IN BATCHES AND HAVING NO INTERMEDIATE BUFFER

ABSTRACT

In this chapter we have studied discrete time transient analysis priority tandem queueing system with two stations in series. K type of customers arrive in batches of random size when all the customers in the batch belong to the same priority class and interarrival time of batches are distributed as exponentially. There is no intermediate buffer between the two stations, which may cause blocking at the first station.

Customers are served in a non-preemptive priority order. The expected delay in the system for the first member of a group of each type of customers is obtained, where all the customers have the same service time distribution. These type of queueing models are important in modelling and analysis of manufacturing systems and computer and co;onnuncation networks.

INTRODUCTION :

Priority queues have important applications in modelling and analysis of manufacturing systems, computer systems and communication networks. Although, a lot of work has been done on priority queues, but, less attention is paid on priority tandem queues.

In this chapter we consider a discrete time transient analysis .tandem priority queueing system with two stations in series having no buffer between the stations, and K type of customers arriving in batches at the system according to discrete time homogeneous poisson processes. The lack of buffer between stations may cause blocking, because a customer finishing its service at the first station cannot move to the second station, if it is busy. Therefore the first station cannot start serving other customers until there is service completion at the second station.

There has been an increased interest in tandem queues with blocking because of their wide applicability, see for example, Dattatreya (1978), Jun & Perros (1990), Xinltchao (1994) Grassman & Dekic (2000), However, no work has been done on priority tandom queue with blocking, having arrival in bulk. The model considered in this chapter is very common in communication system. For example in telecommunication system, message, are transmitted through the network. There is a finite buffer at each station and two types of message regular messages and acknowledgement messages. Acknowledgements messages carry information, about the stations of the system and have priority over regular messages. In packet switching network the message are decomposed into smaller pices called packet each of which has a maximum length so a message can be throught as a group of customers, when packets represent the customers.

There is extensive literature on priority queues see Jaiswal (1968), Kleinrock (1976) and Wolff(1989) for extensive treatment and reference.

MODEL DESCRIPTION:

In this model, we consider two station in series, with no intermediate buffer. We assume that customers arrive in batches of random size according to a discrete time homogeneous poisson process at the first station, with arrival rate λ_i (Number of groups/unit) for priority i customers (i = 1, 2 .... k). It is supposed that all customers in a batch classes are independent of each other. At station one, customers are surved singly in a non preemptive priority order with type i batch having priority over type j batch if i < j, within the same type the order of services is FCFS, within the same group. The order of services in random order i.e. batches of same priority are served in order of their arrival.

When a customer's service is completed station one it moves to station two provided station two is empty at that time i.e. arrival to station two is in single, if station two is busy the customer will wait at station one and block it until station two becomes empty no customer can enter service at station one when it is being blocked.

We assume that the service requirement of type i customers at the first station are i.i.d., r.v's Si with distribution function $F_m(i)$ and density function $f_m(i)$. The station two all the customers have service time requirement T which are i.i.d., r.v's with distribution function G_m. and density function g_m.

From the system structure, it is obvious that the order in which the customers are served at the first station is the same as that of the second station. The amount of time a customer spends at station two is exactly its service time. Hence if we can compute the expected time a customer spends at station one, we know the expected amount of time the customer spends in the whole system.

The batch size X_i is a random variable representing the number of customer in the group of priority i customer.

$P(x_i = n) = g_{i,n}$ $i = 1, 2,\ldots\ldots k$, $n = 1, 2\ldots\ldots$

$E(X_i) = a_i$ and a is mean number of customers in an arbitrary batch i.e. $a = E(a_i)$.

$$\lambda = \sum_{i=1}^{k} \lambda_i \text{ and}$$

$$\lambda_\iota = \lambda_i \sum_{n=1}^{\infty} g_{i,n}$$

we define $Pi = \dfrac{\lambda_i}{\lambda}$

ANALYSIS :

We define the first station as busy if either, the server is serving a customer or is being blocked. If a customer I arrives at a busy station, the time from initiation of its service at station one to the moment of its departure to station which also called the occupation time of a customer at station one is :

$$U_i = \max \{s_i , T\}$$

because all customers have i.i.d. service requirements at the second station the occupation times of first customer at the first station are independent. How ever, type i customer arrives at an empty station one the amount of time the customer stays their

dependents on the remaining service time at station two and its own service requirements at the first station.

Therefore if only the first station is concerned it is exactly $M^X/G/1$ priority queueing system with exceptional service time for the first customers of each busy period.

Now we employ the work conservation approach to analyse the average delays in the system. We have assume that S(i) has distribution function $F_m(i)$ and density function $f_m(i)$ but it a batch of type i arrives an empty system, service requirements of its first customer is different from that of a regular customers in has the same distribution as T_{i0} with distribution function $F_m(i0)$ and density function $f_m(i0)$ where i = 1, 2......k. Therefore the system under consideration is be having like in $M^X/G/1$ queue at station one with a compound poisson arrival process of rate λ an in exceptional service time for customers who arrive at an empty system, because in order to find the average amount of work in a system we treat all the customers as being at the same type (With probability $P_i = \lambda_i/\lambda$) which is a type i customers, now consider an arbitrary batch who arrives at a busy system the service requirement T of the first customer of this batch has distribution function F(m) and density function f(m) where

$$F_m = \sum_{i=1}^{k} F_m(i) \qquad P_i = \frac{\lambda_i a_i}{\lambda a}$$

$$F_m = \sum_{i=1}^{k} F_m(i)$$

$$\text{hence } E(T) = \sum_{i=1}^{k} P_i E(T_{io}),\ E(T^2) = \sum_{i=1}^{k} P_i E(T_{io}^2), \qquad \text{..............(5.1)}$$

Is a customer arrives at an empty system, he receives an exceptional service time T_0 with distribution function $F_m(0)$ and density function $f_m(0)$, where

$$F_m(0) = \sum_{i=1}^{k} P_i F_m(io),\ F_m(0) = \sum_{i=1}^{k} P_i f_m(0)$$

$$\text{Hence } E(T_0) = \sum_{i=1}^{k} P_i E(T_{io}),\ E(T_0^2) = \sum_{i=1}^{k} P_i E(T_{io}^2) \qquad \text{.............(5.2)}$$

The average amount of work in the system, which is also average delay for an arbitrary customer is given by :

$$E(V) = \frac{\lambda a E(T_0^2) + [\lambda^2 a E(T_0) E(T^2) / 1 - \lambda a E(T)]}{2[1 - \lambda a E(T) + \lambda a E(T_0)]} \qquad \text{...........(5.3)}$$

Where $E(T)$, $E(T^2)$, $E(T_0)$ and $E(T_0^2)$ are given by (1) and (2). The proportion of time the server is idle then P_0 is given by :

$$P_0 = \frac{1 - \lambda\, a E(T)}{1 + \lambda a E(T_0) - \lambda a E(T)} \qquad \text{...........(5.4)}$$

Now we assume that S_i is the service time of an arbitrary type i customer and S is the service time of an arbitrary customer, then

$$E(S) = \sum P_i E(S_i),\ E(S_i) = P_0 E(T_{io}) + (1 - P_0) E(T_i)$$

$$E(S_i^2) = P_0 E(T_{io}^2) + (1 - P_0) E(T_i^2),\ I = 1, 2, \ldots\ldots\ldots k,$$

The average amount of work in service is :

$$\sum_{i=1}^{k} \lambda_i g_i P_0 E(T_{io}^2) + (1 - P_0) E(T_i^2)/2$$

Where P_0 is given in eq.n (4) ....(5.5)

We first analyse, Type 1 customers, the delay of the first customer of type 1 batch is the sum of the service times of types of type 1 customers waiting in the queue, and the remaining service time of any customer a type 1 batch, then

$$d_1 = \lambda_1 a_1 d_1 E[T_1] + \lambda a E[S^2] / 2$$

$$= \rho_1 d_1 + \lambda a E[s^2] / 2,$$

Where $\rho 1 = \lambda_1 a_1 d_1 [T_1]$ therefore

$$d_1 = e_1 d_1 + e_2 d_2 + \lambda a E[s^2] /2$$

Then we analyse Type 2 customers. Let R be the amount of work in the system found by a randomly selected customer from an arriving Type 2 batch, the R is the sum of service times of all Types 1 and 2 batches waiting in the queue, plus the amount of work currently in service. Hence

$$E(R) = \lambda_1 a_1 d_1 E[T_1] + \lambda_2 a_2 d_2 E[T_2] + \lambda a E[s^2]/2$$

$$= \rho_1 d_1 + \rho_2 d_2 + \lambda a E[s^2]/2$$

The expected delay of first customer of type 2 batches is the sum of E [R] and the expected service times of new Type 1 arrivals before the beginning of their service, which is $d_2 \lambda_1 a_1 E[T_1]$,

$$d_2 = E[R] + d_2 \lambda_1 a_1 E[T_1] = E[R] + \rho_1 d_2$$

$$\text{hence } d_2 = \frac{\rho_1 d_1 + \lambda a E[S^2]/2}{1-\rho_1-\rho_2} = \frac{\lambda a E(S^2)}{2(1-\rho_1)(1-\rho_1-\rho_2)}$$

$$= \frac{\sum_{i=1}^{k} \lambda_i a_i [\rho_0 E(T_{io}^2) + (1 - P_0) E(T_i^2)}{2(1 - \rho_1)(1 - \rho_1 - \rho_2)}$$

In general, let d_1 be the average delay of the customer of a batch of i then, we have

$$\text{hence } d_i = \frac{1/2\, E(S^2) + \sum_{i=1}^{k} \lambda_j a_j E[Tj]}{1 - \sigma_i}, i = 1, 2 \ldots\ldots k$$

$$\text{Where} \qquad \sigma_i = \sum_{j=1}^{i} \rho_i$$

Therefore, the average delay for a type i customer is obtained as :

$$d_i = \frac{\lambda a E[S^2]}{2(1 - \sigma_{i-1})(1 - \sigma_i)}$$

$$d_i = \frac{\sum_{i=1}^{k} \lambda_i a_i [\rho_0 E(Tio^2) + (1 - P_0) E(T_i^2)}{2(1 - \rho_1)(1 - \rho_1 - \rho_2)}, i = 1, \ldots\ldots k \qquad \ldots\ldots..(5.6)$$

Where $\rho_i = \lambda_i a_i E(Ti)$, and P_0 is given in (4).

We define the first station as busy if other the server is serving a customer, or a customer is being blocked. If a batch of i customer arrive at a busy system, their for the first customer of the batch. The time from the initiation of its service at station one to the initiation of its service at station two, will be called as the occupation time of this customer at station one, which

$$U_i = \max \{s_i, T\} \text{ ; customers}$$

$$F_m = f_m(i)\, G(m) + F_m(i)\, g(m),\ i = 1, 2, \ldots k.$$

However, if a group of i customer arrives at an empty station one, the amount of time the first customer of the batch stays there depends on the remaining service time at station two (if any) and its own service requirement at the first station.

Assume that the first customer of the arrives at the first station units of time after the previous customer departs from the first station, and is a type i customer, then, the amount of time this customer stays at station one in max $\{S_i, T - s\}$. Hence the conditional density function of the amount of time the first customer of type i stays in station one is

$$F_m(i)\, G(m+s) + f_m(i)\, g(m+s).$$

The probability density that a customer arrives at time s and is of class i is $\lambda_i e^{-\lambda s}$, where again $\lambda = \sum_{i=1}^{k} \lambda_i$. Therefore, given that a type i customer arrives at an empty station one, the density function of the amount of time this customer stays in station one is

$$F_m(io) = f_m(i)\, G(m) + e^{\lambda i} (\lambda\, [F_m(i) + f_m(i)] \int e^{-\lambda s} g(s)\, ds$$

$$i = 1, \ldots k \qquad \ldots\ldots\ldots\ldots(5.8)$$

Now, we have obtained that the expected delay of the first customer of a type I batch at the first station is d_i

$$d_i = \frac{\lambda a E\,[S^2]}{2(1 - \sum_{j=1} \rho_i)\,(1 - \sum_{j=1} \rho_i)}$$

Where $\rho_i = \lambda_i E[U_i]$, $\lambda E[S^2] = \sum_{i=1}^{k} \lambda_i a_i [P_0 E(U_{io}^2) + (1 - p) E(U_i^2)$ and $E[U_i]$, $E[U_i^2]$, $E[U_{io}]$ and $E[U_{i0}^2]$ are computed from (5.7) and P_0 is given by

$$P_0 = \frac{1 - \sum_{i=1}^{k} \lambda_i a_i E[U_i]}{1 + \sum_{i=1}^{k} \lambda_i a_i E[U_{i0}] + \sum_{i=1}^{k} \lambda_i a_i E[U_i]} \qquad \text{..........(5.9)}$$

The expected time, the first customer of a type i batch spends at the first station is

$$d_i + P_0 E[U_{i0}] + (1-P_0) E[U_i], i = 1,.....k,$$

and the expected total time, the first customer of a type i batch spends in the whole system is

$$d'_i = d_i + P_0 E[U_{i0}] + (1-P_0) E[U_i] + E[T], i = 1,.....k, \qquad \text{.........(5.10)}$$

Where E[T] is the common mean of the service requirement at station two. From little's formula, the expected number of type i customer in the system

$$L_i = \lambda_i a_i d_i = \lambda_i a_i [d_i + P_0 E(U_{io}) + (1-P_0) E(U_i) + E[T], \quad i = 1, ...k$$

CHAPTER - 6
BIBLIOGRAPHY

BIBLIOGRAPHY

1. Akyildiz I. F. van Bround . H.(1989) : Computational algorithms for networks of queues with rejection blocking. *Acta, infarmatica* 559-576.

2. Altiok, T.(1982) : Approximate analysis of exponential tandem queues with blocking. *EJOR*,390-398.

3. Altiok, T.M. and H.G.Perros(1986) : 'Open network of queens with bloking split and merge'. Configration *IEEE Tran* 18,251-261

4. ALTIOK,Tand H.G.PERROS.(1985):Approximate analysis of arbitrary configurations of open queueing networks with blocking. *Annals of operations research* (to appear)..

5. ALTOIK.T.and perros,H.G.(1987):Approximate analysis of arbitrary configuration of open queueing networks with blocking. *Annals of operations research* 9, 481-509.

6. Ammar M.H.(1980):Modelling and analysis of unreliable manufacturing assembly networks with finite storages. MIT Laboratary for information and decision systems repoart *LIOSTH*-1004.

7. Ammar, M.H.and S.D. Gershwin,(1981) : Equivalence relation in queueing Models of manufacturing.' *Proceeding of the nineteenth IEEE conf. On decision and control.*

8. Avram, F.Bertsimas ,D.and Ricord, M.(1995):Fluid models of sequencing problems in open queueing networks .An optimal control approach.In stochastic networks –IMA volumes in Mathematics and its applications editors, *newyark.*

9. Baba, Y. (1996)-A bulk service G1/M/1 queue with service rates depending on service patch size open res. *Soc, Jpm*. J9-1pp 25-34

10. Balsamo, S, Denitto- Pesone,V.(1994)- A servey of product from queueing network with bloking and their equivelence *Annals of open. Res.* 48,31-61

11. Baskett, F Chandy ,K.M.,Muntz,R.R.and Polacius F.G.(1975) : Open ,closed and mixed network of queues with different classes of customers. *Journal of the ACM*, 22 ,248-260.

12. Bitran, G.R.,Dasu S.(1992):A Review of open queuing networks models of manufacturing system .*Queueing system* 12,95-134.

13. Bitran, G.R. Tirupati, D.(1988):Multiproduct queueing network with deterministic routing decomposition approach and the nation of interference : *Management science* 34(1):75-100.

14. Boxama, O.J (1979) : "On a tandem queueing model with identical services time at both counters. *Adv . in Appl porbad* : 616-659.

15. Boxma, O.J (1982) : Two identical communication channels in series with afinite intermediate buffer and over flow Pre print, university of Utrecht, *dept of Mathematics*.

16. Barke, P.J.(1956) : The output of a queueing system operation research , vol, 4. pp.609-704.

17. Brandwajn, A. Sahai, A.K (1989) : A note on approximate itrative solution of open tandem networkes with blocking performance evaluation ;10-211-218 .

18. Brandwajn, A. and Y.L. Jow (1985) : An approximate melhod for Tandem queues with blocking due to finite waiting Room . *IEEE, Tran, Soft Eng SE*-5. 631-642.

19. Brandwance, A and Jow, Y.J.J. (1988) : "An approximation method for tandem queues with blocking" *operation Research* 36; pp 73-83

20. Burke, R.J. (1968) : The dependence service in Tandem M/M/S queues, *Oper, Res.* 17(754,755)

21. Burke,R.J. (1980) : The dependence of sojourn times in tandem M/M/S queues, J.Assoc. *compute. Mach*, 97,pp 981-986.

22. Buzen, J.P.(1973) : Computational Algorithms for closed queueing networks with exponential *servers, comm.*: ACM 16(527-531)

23. Buzzacott, J. A and D.DYAO (1986) : "On Queueing network models of flexible manufacturing system. *Queueing system theory*. Appl. 1, 5-27.

24. Carmelila Goerg (1985) : Improving mean delay in data communications networks by new combined strategies. *Based on the principle*. Pp. 931-937.

25. Chakka, R.C and I Mitrani (1996) : "Approximate solution for open networks with break downs and repairs in –stochastic netwarks . *Theory and applicalions*.

26. Chandy, K.M. Herzog U and L. Woo (1974) : "Approximate analysis of general queueing network" IMB ; Res .Rep. RC. 4931 *Yorktown Heights NY.*

27. Chaudhry , M. L. and J.G.C Templeton (1983) : A first course in bulk queues , *wiley , New York.*

28. Chaudhry M.L. (1992) : QROOT sotware package. A and A publications, 195 Carrie Crescent, Kingston, ontorio. *KTM 5X7, Canada.*

29. Chaudhry, M.L. Agarwal, M. and Templeton J.G.C. (1992) : Exact and approximate numerical solution of steady –state Distribution arising in the queue G/G/1 to 105-152.

30. Chaudhry, M.L. Kapur P.K.Tempeton J.G.C.(1991) : Discrete Time Transient solution for a class of problems In queueing theory: *Geom(n)/Geom(n)/r/N communicated.*

31. Chaudhry,M.L, Kapur P.K, Tempeton J.G.C.(1977) : A note on the distribution of a busy period for M/M/C queueing system, Math, oper. *For sch. Statist 'H'* (75-79)

31. Chaudhry,M.L. Kapur P.K. Sim, S.H.(1991) : Numerical Computations of Transient solution in Discrete Time for a class of problems in reliability theory. *Communicated.*

32. Chaudhry, M.L B,R. Madill G. Briese (1987) : Computatioinal analysis of steady state probabilities of M/G (a.b) /1 and related non bulk queues. *Queueing system* 2, 93-114.

33. Christophar Floros (1990) : "Diffusion approximation for computer Communication" *network Val.* 31, pp 83-124

34. Chaudhry M.L. and U.C gupta (1999) : Modelling and analysis of M/G (a .b) /1/N queue simple alternative approach .

35. Cohen ,J.N (1956) : "n the queueing percosses of lanes , *Philips Tech .Rept.*

36. Daduna, H. and R. Szekli (2000), " On the correlation of sojourn times in open networls exponential multi server queues, *queueing system*, 34,169-181.

37. Dai .J,G (1995) : "Stability of open muticlass queueing networks via fluid models. In stochastic Network IMAVOLUME in Mathematics and its applications. F. P.K elly and R.J Williams (eds) 71 springer –verlog new york 71-90.

38. DAI J.G. and W. DAI (1999) : A heavy traffic limit theoram for a class of open queueing networks with finite buffers Queueing system, 32. pp 5-40.

39. Dallery Y. (1990) : "Approximate analysis of generalize open queueing networks with restricted capacity. *Performance. Evaluation*, 1,209-222.

40. Dallery Y.Gershwin S.B.(1992) : Manufacturing flow line system a review of models and analytical results.*Queueing system* 12 : 3-94.

41. Dallery, Y. Frain Y. (1993) : "On decomposition methods for tandem queueing networks with blocking" *opr* 41,386-399.

42. Dattatreya E.S.(1978) : Tandem queueing system with blocking Ph. D.Thesis Deparment of industrial Engineering and operation research university of California Barkeley.

43. Disney R.L. and Koning, D.(1985) : "Queueing networks : *A servey of their random processes*." SIAM review 27,335-403.

44. Erramilli, A, W. Willinger and P. Prithi (1994) : "Fractful traffic flows in high speed communication networks. *Fractal 2* (3): 409-412.

45. Fakinos D.(1982) : "The generalized M/G/K blocking system with heterogeneous servers." *opns Res. Soc.* 33, 801-809.

46. Finch P.D. (1959) : The out put process of the queueing system M/G/1,1.*Roy statist soc.B.*vol.21,No.2,pp.375-380.

47. Flores ,C.(1985) : Diffusion approximate for computer communication networks .In computer communication proc.symp.Appl. math.vol .31,B Gopinath (ed), American Mathematical society ,*providence* ,RI.83-124.

48. Foss, S.(1989) : Some properties of open queueing networks. *prob.Inf. Transm.*25 (3),90-97.

49. Fraker,I.R.(1971):Approximate techniques for the analysis of tandem queueing system .Ph. D.dissertation *Dept .of I.E,Clemson university.*

50. Friedman, H.D. (1965) : "Reduction methods for tandem queues" *Oper Res*, 13,121-131.

51. Frost,v.and B. Melamed (1994) : Traffic modelling for telecommunications network .IEEE communications magazine 32,70-80. Control of a multiclass flexible queueing system. *Operations research* vol.45,no.5,677-693.

52. Ganz,O,(1984) : Analysis of multiple bus synchronous and asynchronous communication system performance .*Evel pp.*1-13.

53. Gelenble, E. and G. Pujolle (1976) : The behaviour of single queueing general queqeing network , *Acta informatica* vol. 7-123-136.

54. Gerasimov, I. Alexander(1995) : "On normalizing Constants in multiclass queueing networks." *Operations research*, Vol.43,no.4, P.P. 7-4-715

55. Gershwin, S.B. (1987) : "An efficient decomposition method for the approximate evaluation of tandem queues with finite storage space and blocking" *operation research*, 35(2) PP. 291-305.

56. Gershwin, S.B. and I.C. Schick (1983) : "Modelling and analysis of a batch service queue arising out of manufacturing and system modeling. *Queueing system*, 14. 413-426.

57. Gold H. and P. Tran-Gia (1993) : Performance analysis of a batch service queue arising out of manufacturing and system modelling. *Queueing system*,14, 413 – 426.

58. Greenberg, B.S. (1986) : "Queueing system with returining customers and the order of tandem queues. *Ph.D, Thesis Berkely.*

59. Grassman, W,K, and S. Dekic (2000) : "An analytical solution for a tandem queue with blocking queueing system 36,pp.221-225.

60. Gross, D and Harris, C.M. (1985) : Fundamentals of queueing theory , second edition , *john wiley*, New York.

61. Gun, L. and makowski, A.M. (1989) : "An approximation method for general tandem queuing system subject to blocking." *Queueing network with blocking H.G. Perros and T.Altiok* (eds) p.p. 147-171.

62. Gupta D. and Buzacott ,J.A. (1990) : "Production system with two job classes and change over times and revisitation .' Q*ueueing system* 6, 353, 368.

63. Harrison J.M. and Williams R.J.(1987) : Brownian models of open Queueing network with homogeneous customer populations stochastics,22,77-115.

64. Harrison, J.M. and Wein L. M.(1989) : Scheduling network of queues heavy traffic analysis of a simple open network Queueing system theory and application .5,265-280.

65. Harrison J.M. (1978) : The diffusion approximate for tandem queues in heavy traffic advances in applied probility.10.

66. Harrison J.M. and Nguyen .V.(1993) : Brownian models of multiclass queuing netrwork: current status and open problems. *Queueing system*,13,5-40.

67. Harrison P.G.(1985) : "On normalizing constants in queueing network." *Opens Res*. Vol. 33.no2, 464.

68. Harrrisom J.M. and Nguyen V. (1990) : "The QNET method for two moment analysis of open queueing networks. *Queueing system*, 6, 1-32.

69. Harry G.Perros (1990) : Approximate algorithms for open queueing network with blocking .P.P.451-499.

70. Harvath L.(1992) : "Strong approximations of open queuing network mathematics of *operations research.*" 17,487-508.

71. Helber, S. (1988) : Decomposition of unreliable assembly disassembly network with limited buffer capacity and random processing times European jounnal of *operational Research* 109 : PP.24-42.

72. Hillier, F.S and Boling, R.W. (1976) : Finite queues in series with exponential or Erlang service times. *A numerical approach open Res*.15.286,303.

73. Hunter, T.T. (1983) : Mathematical technique of applied probability. Vol. I-discrete time models basic theory academic press New York .

74. Iliadis, I. (1992) : "Synchronous versus asynchronous operation of a packet switch with combined input and output queueing" *Performance Evaluation*, 16, pp. 241-250.

75. Jackson, J.R. (1957) : Network of waiting lines, *operations research*, Vol. 5,no.4.

76. Jain, S and Smith, J. M. (1994) : "Open finite queueing networks with M/M/CK parallel servers." *Computer and operations research* 21 (3) 297-317.

77. Jun, K.P. and perros, H.G.(1990) : "An approximate analysis of open tandem queueing networks with blocking and general service times." *EJOR*, 46, 123-135.

78. Jenq, Y.C. (1984): "Approximation for packetized voice in statistical multiplexer process." *proc. INFOCOM*, 2 56-259.

79. Jung shyrwiv (1992) : "Maximum entropy analysis of open queueing network with group arrival" *J. oper. Res. Soc.* Vol. 43 no11, pp. 1063-1078.

80. Kapur, P.K., Garg R.B., Sehgal V.K., Mishra, G.D. (1996) : "Consider discrete time transient solution for a first passage time analysis, *international journal of management and system (IJOMAS)* Vol. 12, No, 3 Sept-Dec (1996).

81. Kapur P.K., Garg R.B.., Sehgal V.K., Mishra G.D. (1996) : "Give numerical computations of discrete time solutions for a multi server queue with bulting renging: *Asia pacific journal of operational research*, Vol. 13, Page 1-15.

82. Kapur P.K., Sehgal , V.K. and Garg, R.B. (1995) : "Numerical and transient solutions of K out N.G. systems in discrete time with heterogeneous repair facilities.

83. Kapur P.K., Sehgal V.K., Garg R.B., Jha A.K. (1995) : "Give numerical and transient solutions of K out N:G systems in direcrete time with multi repair facilities, *Bulletin of applied sciences section*, E, (Mathematics).

84. Kella,O, and whitt. W. (1992) : A Tandem fluid network with Levy input. In queueing and related models (ed. U.N. Bhat and I.V, Basava). *Oxford university press*, 112-128.

85. Kelly, F.P. and laws, C.N. (1993) : "Dynamic routing in open queueing networks: Brownian models, cut constrains and resource pooling queuing system: *Theory and applications* 13, 47-86.

86. Kendall, D.G. (1951) : Some problems in the theory of queues Jour, Roy. *Soc, B*-13, page 151-173.

87. Kerbache L. smith J.M. (1988): "Asymptotic behaviour of the expansion method for open finite queuing networks." Computer and operations research, 15(2): 157-169.

88. Kerbache L. smith, J.M. (1987) : The generalized expansions method for open finite queuing networks. *European journal of Operational Research*, 32,pp. 448-461.

89. Kerbache, L. and J.M. smith(1986) : The generalized expansion method of open finite queuing networks. Technical report, Dept of Industrial Engineering and operations research, *Univ. of Massachusetts Amherst, Mass.*

90. Kijima m. (1992) : "The transient solutions of class of markovian queues. *Computers and mathematics with applications* 24, 17-24.

91. King, J.B. Potes(1998) : "Queueing models for buffers with deal-up servers." *Performance evaluation* -10, 129-145.

92. Klienrock (ed.) (1998) : "Queueing system,vol.-II Computer applications." *John wiley and sons, New York.*

93. Knessi, C. and C. Teir (1990) : "Two tandem queues with general input." *I. SIAMJ appl. Math*, 59, pp, 19171959.

94. Kobayashi, H.(1974) : "Application of the diffusion approximation to queueing networks : Equilibrium queue distributions"*journal of the ACM* 21(2), 316-328.

95. Kobayashi H. (1983) : "Discrete time queueing system. In probability. Theory and computer science. Louchard, G. Latouche, G. (eds), *Academic press New York.*

96. Kuehn, D.J. (1979) : "Approximate analysis of generate networks by decomposition ." *IEEE. Transactions on communication*, 27(1), 113-126.

97. Langaris, C. and conolly, B.(1985) : Three stage tandem Queues with blocking , *European j.oper , Res.* 19,222-232.

98. Latouche, G. and Neuts, M. F. (1980) : "Efficient Algorithmic solution to exponential tandem queues with blocking. *SIAM J.Algebraic discrete method* 1,93-106.

99. Lavenberg, S.S. (1975) : "Staibility and maximum departure rate of certain open queueing networks having finite capacity constraints." *IBM Res. Rep. RJ 1625. San, jose.*

100. Lee H.S, Bouhouch, A. Dallery, Y. Frein Y. (1998) : Performance Evaluation of open queueing networks with arbitrary configuration and finite buffers. *Annals of operations Research*, 79:181-206.

101. Lee, H.S. and Pollock, S.M. (1989) : Approximate analysis for the merge configuration of an open queueing network with blocking. *IIE Transactions, 21,no* 2, pp.122-129.

102. Liv, C.M. and Lin, C.L. (1995) : "An efficient two phase approximation methods for exponential tandem queueing system with blocking." *Computer and operational rsearch* 22 (7) , 745-762.

103. Marshal, W.T. and S.p. margan (1985) : Statistics of mixed data traffic on a local area network, *computer network and ISDN system*, 10: 185-194.

104. Morse P.M. (1958) : Queues, Inventories and queueing theory. John Willey, *New York.*

105. Neutus, M.F. (1968) : "Two queues in series with a finite intermediate waiting room . J. *Appl. Probab.* 5,123-142.

106. Nguyen N. (1995) : "Fluids and diffusion approximation of a two station mixed queueing network. *Mathematics operations research* 20, 321-354.

107. OHNO. Katsuhisa (1987) : "Computing optimal policies for controlled tandem queueing system ." *operation research* vol. 35,no.1 pp. 121-126.

108. Pal, Devendra (2002), " Study of some tandem queueing models with reference to manufacturing process and communication networks." *Ph.D. Thesis Agra.*

109. Park, D.H. perros and H. Yamashita (1994) : "Approximate analysis of discrete time tandem networks with busty and correlated input traffic and customer loss. *Open. Res. Lett.* 15, 95-104.

110. Parthasarthy, P.R. (1987) : "A transient solution to a M/M/1 queue , A simple approach adv. *Appl. Prob*. 19 (977-998).

111. Perros ,H.G. and P.M. Snyder, (1986) : "A computationally efficient approximotion algorithms of analyzing open queueing netwarts with blocking, manuscript, Computer science Department, *North Corolina State University.*

112. Perros .H.G and snyder, P. M. (1989) : A computationally efficient approximation algorithm for feed forward open queueing networts with blocking, *performance evaluation*, 9, 217, 224.

113. Perros H.G. and T.M. Altiok (1986) : An approximate analysis of open network of queus with blocking. *Tandem configuration IEEE trans software, Eng*. Vol. SE-12(3) pp. 450-461.

114. Perros H.G. Liu Y.C. nilsson, A.A. (1988) : "Approximate analysis of product from type queueing network with blocking and dead lock." *Performance evaluation,* 8. 19-39.

115. PERROS, H.G (1981) : Asymmetrical exponential open queue network with blocking and feed back. *IEEE Transaction on software engineering* 7, 395-402.

116. Peterson W.P (1991) : Diffsion approximations for networks of queues with multiple customer types. *Mathematics of operations research* 9,90,118.

117. Pinedo, M(1982) : "On the optimal order of station in tondem queues in applied probability –computer science the interface, vol ,II.R.L. Disney and T.J. ott (Eds), *Birkhauser, Boston,* b, 307,325.

118. Pollok, S.M. and J.M. Alden. (1985) : Approximation analysis for open tandem queues with blocking. Expeonential and general service distributions technical report, IOE. Dept, 85-130, *University of Michigan, ANN, ARBOR.*

119. Pots, G. (1995) : State dependent queueing networks : Approximation and applications, Ph.D. dissertation, Departments of Industrial Engineering and management technion, *Haifa, Israel.*

120. Reich E. (1963) : "Note on queues in tandem *Ann, Math Statist* Volume 34, (338-341).

121. Reich, E., (1957) : "Waiting times when queues are in tandem *Ann. Math, Statist,* 28, (768-773).

122. Reiman, M.I. (1984) , " Open queueing networks in heavy traffic. *Math oper. Res.* 9, 441-458.

123. Reiser, M. (1979) " A queueing network analysis of computer communication networks with window flow control" *IEEE. Transactions on communications* 8, 1199-1209.

124. Reyravi, Hassan (1993) : "Limiting distribution of the blocking probability for circuit switched networks" *Computer Elect. Eng.* Vol. 19, No. 3, pp, 231-244.

125. Rubin , I (1974) : Path dec ays in communication networks " *Appl. Math optimiz*, Vol.1.

126. Rybko, A.K. and Stolyar , A.L. (1991) : "Ergodicity of stochastic process describing the operation of an open queueing network problemy Peredachi infrasnatsii, 28, 2-26.

127. Sakasegawa H. Kawashima, T. (1994) : On the equivalence of three types of bloking in non – markovian tandem queues. *Queueing systems* 16 : 301-3.

128. Sauer, C.H.(1981) : Approximate solution of queueing networks with simultaneous resource possession IBM.J res. *Devlopment* 25,894-903

129. Schmidt Valker (1987) : "On joint queue length characteristics in infinite server tandem queues with heary traffic." *Advance appl. Prob.* 19. 474-486

130. Schweitzer, P. (1976) : "Maximum through put in finite capacity open queueing networks: IBM Res. R.C. 5996. *Yarktown Heights*. N. Y .

131. Seraphin B, Cab (1981) : Message delays in repeated service tandem connection'' *IEEE trans communication* 670-677.

132. Shanti kumar ,J. G. M.A. Jafari, (1987) : Bounding the performance of tandem with finite buffer spaces working paper. *University of California at Berkeley.*

133. Shanti kumar ,J.G.and J.A.buzacott (1981) : "Open queueing network models of dynamic jobs hops internate . *J. production Res.*19,3,255-266.

134. Sharma, O.P. and Dass S. (1988) : "Multiserver markovain queues with finite waiting space". *Sakhya*, B.50. 328-331.

135. Sharma.O.P. (1990) : Markovian queues ELLISHORWOOD.

136. Sigman, K. (1990) : "The stability of open queueing networks Stochastic processes and application." 35,11-25.

137. Skianis, C.A. Kouvatsos D.D.(1998) : Arbitrary open queueing networks with server vacation periodes and blocking. *Annals of operation research* ,79 : 143-180.

138. Spiros, G. Dnozis (1993) : Entropy Maximized queueing networks with blocking and multiple job clases. *Performance Evaluation* 17, 189-205.

139. Srivastava.H.M and Kashgap,B.R.K.(1982) : Special fuctions in queueing theory and related sto chastic processes *academic press New York* John wiley New York

140. Strelen, J.C. Bark B. becker ,J. Jonas V. (1979) : "Analysis of queueing network with blocking using a new aggregation technique *annals of operation research* ,121-142.

141. Tackaces L. (1962) : Introduction to the theory of queues oxford university press *New York.*

142. Takahashi, Y. Miyahara, H. Hasegawa, T. (1980) : "An approximation methods for open restricted queueing networks *oper. Res*,. 28 (3), Part I, 594-602.

143. Tembe,S.V.and R.W. wolff .(1974) "The optimal order of service in tandem queues." *Open .Res.*24,824-832.

144. Tsouces, P. and Walrand, J. (1987) : "On the interchangeability and stochastic ordering of /M/I/ queues in tandem." *Adv. in appl. Probab*, 19, 515-520.

145. Van Dijk, N. and J. Vanderwal (1989) : "Simple bounds and monotonicity results for multi server exponential tandem queues queueing system, 4, pp, 1-16.

146. Victor A. Gvnitshi (2001) : "Networks of simple server queues with dependent service times" *Queueing system* 37, pp, 363-377.

147. Vijaya Laxmi and U.C. Gupta (1999) : "On the finite buffer balk service queue with general independent arrivals G1/M (b)/1/N. oer. *Res. Letters*, 25, 241-245.

148. Vijaya Kumar, C.V.R.S (1996). "Some interdependent tandem queueing models with applications to communications networks." *Ph.D. thesis Andhra University Visakhapatnam India.*

149. Vinod, B. and Altoik, T. (1986). " Approximating unreliable queueing networks under the assumption of exponentiality. *Journal of the operations Reasearch Society*, 37. 309-316.

150. Waller, A. Aw. (1994). " A queueing network models for field service support system." *Omega Int. I. Mgmt*. Vol. 22 35-40.

151. Weber, R.R. (1979). "The Interchangeability of tandem./M/1 Queues in serives." *J. appl. Probab*. 16, 690-695.

152. Whitt, W. (1983). "The queueing network analyzer." *Bell, sys. Tech. J.* 62 (9) pp. 2779-2815.

153. Williams, R.J. (1998). "Diffusion approximations far open multicless queueing networks : Sufficient conditions involving state space collapase, *Queueing systems*. 30, pp. 21-28.

154. Wolff, R.W. (1982), " Tandem queues with dependent service in light traffic. *Oper, res*, 30, 619-635.

155. Xinltchao (1994). "A priority tandem queue with no intermediate Buffer." *J. opes. Res. Soc.* No. 3, pp. 321-329.

156. Xinltchao, (1992) : "Reversibility of tandem queues with blocking." *Naval Research Logistics* Vol. 39, pp 957-974.

157. Yadin, M. Naor, P. – Queueing system with a removal service station. *Operational Research Quarterly* 1963 : 14: 393-405.

158. Yakuo Hyshida (1993). " Throughput analysis of tandem type goback-NARQ Scheme for satellite communications." *IEEE Trans. Communications*. 41, 1517-1524.

159. Yannopoulose, Alfa, A. S. (1994). A simple and quick approximation algorithm for tandem split and merge queueing networks with blocking *INMFOR*, 32 (4) : 288-308.

160. Yao, D. D. Buzacott, J. A. (1985). Modelling class of state dependent routing inflexible manufacturing systems. *Annals of operations research* 3, 153-167.